Analysis of the periodically fragmented environment model : II - Biological invasions and pulsating travelling fronts.

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Abstract

This paper is concerned with propagation phenomena for reaction-diffusion equations of the type

$$u_t - \nabla \cdot (A(x)\nabla u) = f(x,u),\ \ x \in \mathbb{R}^N$$

where A is a given periodic diffusion matrix field, and f is a given nonlinearity which is periodic in the x-variables. This article is the sequel to [8]. The existence of pulsating fronts describing the biological invasion of the uniform 0 state by a heterogeneous state is proved here. A variational characterization of the minimal speed of such pulsating fronts is proved and the dependency of this speed on the heterogeneity of the medium is also analyzed.

Contents

1 Introduction and main results

This paper deals with the mathematical analysis of a periodically fragmented environment model which is given by the reaction-diffusion equation

$$u_t - \nabla \cdot (A(x)\nabla u) = f(x, u), \quad x \in \mathbb{R}^N \tag{1.1}$$

with periodic dependence in the x variables. It is the sequel to the paper [8], which focused on some conditions for the stationary equation

$$\begin{cases} -\nabla \cdot (A(x)\nabla p) = f(x, p) \ \text{ in } \mathbb{R}^N, \\ p(x) > 0, \ x \in \mathbb{R}^N, \end{cases} \tag{1.2}$$

to have a bounded solution, and on the effects of the heterogeneity in x. The present paper is concerned with the propagation phenomena, and especially the propagation of fronts, associated to (1.1) – precise definitions will be given below. Some formulas for the speeds of propagation of fronts are proved and the dependence in terms of the coefficients of (1.1) is analyzed.

The archetype of such reaction-diffusion models is the following equation

$$u_t - \Delta u = f(u) \ \text{ in } \mathbb{R}^N \tag{1.3}$$

which was introduced in the pioneering papers of Fisher [15] and Kolmogorov, Petrovsky, Piskunov [25]. An example of nonlinear term is given by the logistic law $f(u) = u(1-u)$. This type of equation was first motivated by population genetics, and, as (1.1), it also arises in more general models for biological invasions or combustion.

Of particular interest are the propagation phenomena related to reaction-diffusion equations of the type (1.3), or (1.1). First, equation (1.3) may exhibit planar travelling fronts, which are special solutions of the type $u(t, x) = U(x \cdot e + ct)$ for some direction e ($|e| = 1$, $-e$ is the direction of propagation) and $U : \mathbb{R} \to (0, 1)$ (assuming that $f(0) = f(1) = 0$). Such solutions are invariant in time in the comoving frame with speed c in the direction $-e$. Second, starting with an initial datum $u_0 \geq 0$, $\not\equiv 0$ which vanishes outside some compact set, then, under some assumptions on f, $u(t, x) \to 1$ as $t \to +\infty$; furthermore, the set where u is close to 1 expands at a certain speed which is the asymptotic speed of spreading and which, in the case of equation (1.3) with a nonlinearity f positive in $(0, 1)$, is the minimal speed of planar fronts (see *e.g.* [1]).

Whereas the homogeneous equation (1.3) has attracted many works in the mathematical literature, propagation phenomena for *heterogeneous* equations of the type (1.1), where both the diffusion and the reaction coefficients depend on the space variables x, were studied more recently (see *e.g.* [4, 16, 21, 26, 30]). In models of biological invasions, the heterogeneity may be a consequence of the presence of highly differentiated zones, such as forests, fields, roads, cities, etc., where the species in consideration may tend to diffuse, reproduce or die with different rates from one place to another.

One focuses here on *periodic* environments models and for which the diffusion matrix $A(x)$ and the reaction term $f(x,u)$ now depend on the variables $x=(x_1,\cdots,x_N)$ in a periodic fashion. As an example, f may be of the type

$$f(x,u)=u(\mu(x)-\kappa(x)u), \tag{1.4}$$

or even, simply,

$$f(x,u)=u(\mu(x)-u), \tag{1.5}$$

where the periodic coefficient $\mu(x)$, which may well be negative, can be interpreted as an effective birth rate of the population and the periodic function $\kappa(x)$ reflects a saturation effect related to competition for resources. The lower μ is, the less favorable the environment is to the species.

These models for biological invasions in unbounded domains were first introduced by Shigesada et al. in dimensions 1 and 2 (see [23, 26, 27]). In these works, the nonlinearity f is given by (1.5), and A and μ are piecewise constant and only take two values. This model is then referred to as the patch model. Numerical simulations and formal arguments were discussed in [23, 26, 27] about this model – in space dimensions 1 or 2. The various works of Shigesada and her collaborators have been an inspiring source for the present paper. We aim here at proving rigorously some properties which had been discussed formally or observed numerically. The introduction of new mathematical ideas will furthermore allow us to derive results in greater generality and for higher dimensional problems as well.

In the paper [8] and in the present one, we discuss these types of problems in the framework of a general periodic environment, and we give a complete and rigorous mathematical treatment of these questions. In the first paper [8], we discussed the existence of a positive stationary state of (1.1), that is a positive bounded solution p of (1.2). The latter is referred to as biological conservation. We also analyzed in [8] the effects of fragmentation of the medium and the effects of coefficients with large amplitude on biological conservation. Here, we connect the condition for species survival (existence of such a solution p) to that for propagation of pulsating fronts for which the heterogeneous state p invades the uniform state 0. This type of question is referred to as biological invasion. We also analyze the effects of the heterogeneity of the medium on the speed of propagation. We especially prove a monotonous dependence of the speed of invasion on the amplitude of the effective birth rate.

Let us make the mathematical assumptions more precise. Let $L_1,\cdots,L_N>0$ be N given positive real numbers. A function $g\ :\mathbb{R}^N\to\mathbb{R}$ is meant to be periodic if $g(x_1,\cdots,x_k+L_k,\cdots,x_N)\equiv g(x_1,\cdots,x_N)$ for all $k=1,\cdots,N$. Let C be the period cell defined by

$$C=(0,L_1)\times\cdots\times(0,L_N).$$

The diffusion matrix field $A(x)=(a_{ij}(x))_{1\le i,j\le N}$ is assumed to be symmetric ($a_{ij}=a_{ji}$), periodic, of class $C^{2,\alpha}$ (with $\alpha > 0$),[1] and uniformly elliptic, in the sense that

$$\exists \alpha_0 > 0,\ \forall x \in \mathbb{R}^N,\ \forall \xi \in \mathbb{R}^N, \quad \sum_{1 \le i,j \le N} a_{ij}(x)\xi_i\xi_j \ge \alpha_0 |\xi|^2. \tag{1.6}$$

The function $f \ : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}$ is of class $C^{1,\alpha}$ in (x,u) and C^2 in u, periodic with respect to x. One assumes that $f(x,0) = 0$ for all $x \in \mathbb{R}^N$ and one sets $f_u(x,0) := \lim_{s \to 0^+} f(x,s)/s$. Furthermore, throughout the paper, one assumes that

$$\forall x \in \mathbb{R}^N, \quad s \mapsto f(x,s)/s \text{ is decreasing in } s > 0 \tag{1.7}$$

and

$$\exists M \ge 0,\ \forall s \ge M,\ \forall x \in \mathbb{R}^N, \quad f(x,s) \le 0. \tag{1.8}$$

Examples of functions f satisfying (1.7-1.8) are functions of the type (1.4) or (1.5), namely $f(x,u) = u(\mu(x) - \kappa(x)u)$ or simply $f(x,u) = u(\mu(x) - u)$, where μ and κ are $C^{1,\alpha}$ periodic functions.

Let λ_1 be the principal eigenvalue of the operator $\mathcal{L}_0$ defined by

$$\mathcal{L}_0 \phi := -\nabla \cdot (A(x)\nabla\phi) - f_u(x,0)\phi,$$

with periodicity conditions. Namely, λ_1 is the unique real number such that there exists a C^2 function $\phi > 0$ which satisfies

$$\begin{cases} -\nabla \cdot (A(x)\nabla\phi) - f_u(x,0)\phi = \lambda_1 \phi \text{ in } \mathbb{R}^N, \\ \phi \text{ is periodic, } \ \phi > 0. \end{cases} \tag{1.9}$$

One says that 0 is an unstable solution of (1.2) if $\lambda_1 < 0$, and "stable" if $\lambda_1 \ge 0$.

We especially proved in [8] that, if $\lambda_1 \ge 0$, then 0 is the only nonnegative bounded solution of (1.2) and any solution of (1.1) with bounded nonnegative initial condition u_0 converges to 0 uniformly in $x \in \mathbb{R}^N$ as $t \to +\infty$ (one refers to this phenomenon as extinction). On the other hand, if $\lambda_1 > 0$, then there is a unique positive bounded solution p of (1.2), which turns out to be periodic,[2] and the solution $u(t,x)$ converges to $p(x)$ locally in x as $t \to +\infty$, as soon as $u_0 \ge 0, \not\equiv 0$.

The above results motivate the following

Definition 1.1 *We say that the* hypothesis for conservation *is satisfied if there exists a positive bounded solution p of (1.2).*

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[1] The smoothness assumptions on A, as well as on f below, are made to ensure the applicability of some a priori gradients estimates for the solutions of some approximated elliptic equations obtained from (1.1) (see Lemma 2.10 in Section 2.3). These gradient estimates are obtained for smooth (C^3) solutions through a Bernstein-type method, [5]. We however believe that the smoothness assumptions on A, as well as on f, could be relaxed, by approximating A and f by smoother coefficients.

[2] Notice that the periodicity is forced by the uniqueness, but was not a priori required in the formulation of equation (1.2).

A simple necessary and sufficient condition for the hypothesis for conservation (or survival) to be satisfied is that $\lambda_1 < 0$, and the solution p is then unique and periodic. This hypothesis is fulfilled especially if $f_u(x,0) \geq 0, \not\equiv 0$. An example of a function f satisfying the hypothesis is the classical Fisher-KPP nonlinearity $f(x,u) = f(u) = u(1-u)$ (see [15, 25]), where $p(x) \equiv 1$. For a general nonlinearity f satisfying (1.7) and (1.8), comparison results and conditions on $f_u(x,0)$ for λ_1 to be negative are given in [8] (see also Theorem 1.3 below). However, it is not easy to understand in general the interaction between the heterogeneous diffusion and reaction terms.

One focuses here on the set of solutions which describe the invasion of the uniform state 0 by the periodic positive function p, when the hypothesis for conservation is satisfied. A solution $u(t,x)$ of (1.1) is called a *pulsating travelling front* propagating in the direction $-e$ with the effective speed $c \neq 0$ if

$$\left\{\begin{array}{rcl} \forall\ (t,x)\ \in\ \mathbb{R}\times\mathbb{R}^N,\ u_t - \nabla\cdot(A(x)\nabla u) & = & f(x,u),\\ \forall\ k\ \in\ \displaystyle\prod_{i=1}^N L_i\mathbb{Z},\ \forall\ x\ \in\ \mathbb{R}^N,\ u\left(t+\frac{k\cdot e}{c},x\right) & = & u(t,x+k),\end{array}\right. \tag{1.10}$$

with the asymptotic conditions

$$u(t,x) \underset{x\cdot e\to-\infty}{\to} 0,\ u(t,x)-p(x) \underset{x\cdot e\to+\infty}{\to} 0. \tag{1.11}$$

The above limits are understood as local in t, and uniform in the directions of $\mathbb{R}^N$ orthogonal to e.

Our first result is the following existence theorem :

Theorem 1.2 *Under the above assumptions on A and f, and under the hypothesis for conservation, there exists $c^* > 0$ such that problem (1.10-1.11) has a classical solution (c,u) if and only if $c \geq c^*$. Furthermore, any such solution u is increasing in the variable t.*

Lastly, the minimal speed c^* is given by the following variational formula

$$c^* = \min\left\{c,\ \exists\ \lambda > 0 \text{ such that } \mu_c(\lambda) = 0\right\},$$

where $\mu_c(\lambda)$ is the principal eigenvalue of the elliptic operator

$$\begin{array}{rcl} -L_{c,\lambda}\psi & = & -\nabla\cdot(A(x)\nabla\psi) - 2\lambda\ eA(x)\nabla\psi\\ & & -[\lambda\nabla\cdot(A(x)e) + \lambda^2 eA(x)e - \lambda c + f_u(x,0)]\psi,\end{array} \tag{1.12}$$

with periodicity conditions.

Before going further on, let us briefly comment this result and recall some earlier works in the literature. Observe first that the formula for the minimal speed simply reduces to the well-known Fisher KPP formula $2\sqrt{f'(0)}$ for the minimal speed of planar front $\phi(x\cdot e + ct)$ for the homogeneous equation $u_t - \Delta u = f(u)$ in $\mathbb{R}^N$ with f satisfying (1.7-1.8) and $p(x) \equiv \min\ \{s > 0,\ f(s) \leq 0\}$. Periodic nonlinearities $f(x,u)$ in space dimension 1 were first considered by Shigesada, Kawasaki and Teramoto [27], and by

Hudson and Zinner [21].[3] The case of equations $u_t - \Delta u + v \cdot \nabla u = f(u)$ with shear flows $v = (\alpha(y), 0, \cdots, 0)$ in straight infinite cylinders $\{(x_1, y) \in \mathbb{R} \times \omega\}$ was dealt with by Berestycki and Nirenberg [11], under the assumption that f stays positive in, say, $(0,1)$; min-max type formulas for c^* were obtained in [17]. Berestycki and Hamel [4] generalized the notion of pulsating fronts and got existence and monotonicity results in the framework of more general periodic equations $u_t - \nabla\cdot(A(x)\nabla u) + v(x)\cdot\nabla u = f(x,u)$ in periodic domains, under the assumption that $f \geq 0$ and $f(x,0) = f(x,1)$, $f(x,s) > 0$ for all $s \in (0,1)$. A formula for the minimal speed is given in [7] under the assumption that $f(x,s) \leq f_u(x,0)s$ for all $s \in [0,1]$ and the dependence of c^* in terms of the diffusion, advection, reaction coefficients as well as the geometry of the domain, is analyzed. Some lower and upper bounds for the minimal speed when the advection term v is large are given in [2, 3, 6, 12, 19, 24]. Lastly, let us add that some previously mentioned works, as well as other ones, [1, 4, 10, 11, 13, 14, 17, 18, 19, 20, 22, 28, 30], were also devoted to other types of nonlinearities (combustion, bistable), for which the speed of propagation of fronts may be unique.

One of the difficulties and specificities of problem (1.10-1.11) with a nonlinearity f satisfying (1.7-1.8) is that f may now be negative at some points x, whereas it is positive at other places, for the same value of u. Besides the existence of pulsating fronts and the variational characterization of the minimal speed, Theorem 1.2 above also gives the monotonicity of all fronts in the variable t (notice that a similar formula for c^* was given by Weinberger in [29], with a different approach, but the monotonicity of the front was a priori assumed there).

Consider now a nonlinearity f satisfying (1.7-1.8), and such that $f_u(x,0) = \mu(x) + B\nu(x)$, where μ and ν are given periodic functions and B is a positive parameter. If $\lambda_1 < 0$ (namely if the hypothesis for conservation is satisfied), one calls $c^*(B)$ the minimal speed, given in Theorem 1.2, of the pulsating fronts solving (1.10-1.11). The following theorem especially gives a monotonous dependency of $c^*(B)$ on B as well as some lower and upper bounds for large or small B (when $\mu \equiv 0$, B can then be viewed as the amplitude of the effective birth rate of the species in consideration). Furthermore, Theorem 1.3 below also deals with the influence of the heterogeneity of f on the minimal speed of pulsating fronts.

Theorem 1.3 *Assume that A is a constant symmetric positive matrix and assume that f satisfies (1.7-1.8) and that $f_u(x,0)$ is of the type $f_u(x,0) = \mu(x) + B\nu(x)$, where μ and ν are given periodic $C^{1,\alpha}$ functions, and $B \in \mathbb{R}$.*

a) Assume that $\max\ \nu > 0$. *Then the hypothesis for conservation ($\lambda_1 < 0$) is satisfied for $B > 0$ large enough and*

$$c^*(B) \leq 2\sqrt{eAe\max(\mu + B\nu)}\ ;$$

furthermore, if $\mu = \mu_0$ is constant, then $c^*(B)$ is increasing in B (for B large enough so that $\lambda_1 < 0$).

[3] Hudson and Zinner proved the existence of one-dimensional pulsating fronts for problems of the type $u_t - u_{xx} = f(x,u)$, provided $c \geq c^*$, but did not actually prove that c^* was the *minimal* speed.

b) Assume that $\int_C \mu \geq 0$, $\int_C \nu \geq 0$ *and* $\max\ \nu > 0$. *Then the hypothesis for conservation is satisfied for all* $B > 0$, *and* $c^*(B)$ *is increasing in* $B > 0$ *under the additional assumption that* $\mu = \mu_0 \geq 0$ *is constant. Furthermore, for all* $B > 0$,

$$2\sqrt{\frac{eAe}{|C|}\int_C (B^{-1}\mu(x) + \nu(x))dx} \leq \frac{c^*(B)}{\sqrt{B}} \leq 2\sqrt{eAe\ \max(B^{-1}\mu + \nu)}$$

and

$$\frac{1}{2}\sqrt{eAe\ \max\ \nu} \leq \liminf_{B\to+\infty} \frac{c^*(B)}{\sqrt{B}} \leq \limsup_{B\to+\infty} \frac{c^*(B)}{\sqrt{B}} \leq 2\sqrt{eAe\ \max\ \nu}.$$

c) Assume that $\mu \equiv 0$, $f_u(x,0) = B\nu(x)$ *with* $\int_C \nu \geq 0$, $\max\ \nu > 0$. *One has*

$$\lim_{B\to 0^+} \frac{c^*(B)}{\sqrt{B}} = 2\sqrt{\frac{eAe}{|C|}\int_C \nu(x)dx}.$$

Theorem 1.3 implies especially that, when $f_u(x,0)$ is of the type $\mu(x) + B\nu(x)$, and no matter how bad the environment may be elsewhere, it suffices to have a very favorable (even quite narrow) zone (namely $\nu > 0$ somewhere) to allow for species survival and to increase the speed of propagation of fronts. Furthermore, the speed is comparable to $\sqrt{B}$ for large amplitudes B as soon as, say, $\int_C \nu > 0$.

Lastly, call $c^*[\mu]$ the minimal speed of pulsating travelling fronts solving (1.10-1.11) with $f_u(x,0) = \mu(x)$, provided the assumption for conservation is satisfied. From the previous theorem, one immediately deduces the following corollary :

Corollary 1.4 *Assume that* A *is a constant symmetric positive matrix and assume that* f *satisfies (1.7-1.8), with* $f_u(x,0) = \mu(x)$. *Assume that* $\int_C \mu \geq \mu_0|C|$ *with* $\mu_0 > 0$. *Then* f *satisfies the hypothesis for conservation and*

$$c^*[\mu] \geq c^*[\mu_0] = 2\sqrt{(eAe)\mu_0}.$$

This corollary simply means that the heterogeneity of the medium increases the speed of propagation of pulsating fronts, in any given unit direction of $\mathbb{R}^N$.

As already underlined, the main difference with the results in [4], in the existence and monotonicity result (Theorem 1.2), is that the function f here is not assumed to be nonnegative. The nonnegativity of f played a crucial role in [4], where the existence of the minimal speed c^* was proved by approximating f with cut-off functions, as in [11]. Although we solve some regularized problems in bounded domains as in [4], the method used in this paper is rather different since we directly prove that the set of possible speeds c is an interval which is not bounded from above, and we define c^* as the minimum of this interval. Existence of pulsating fronts is proved in Section 2. Monotonicity is proved in Section 3. Lastly, the characterization of c^* is given in Section 4, as well as the effects of the heterogeneity of the medium on the propagation speeds.

2 Existence result

This section is devoted to the proof of the existence of pulsating fronts for (1.10-1.11) for large speed. Throughout this section, one assumes that the hypothesis for conservation is satisfied, namely that there exists a (unique) positive bounded solution p of (1.2), which is periodic.

2.1 Existence result in finite cylinders for a regularized problem

Let us make the same change of variables as Xin [30] and Berestycki, Hamel [4]. Let $\phi(s,x)$ be the function defined by :

$$\phi(s,x) = u\left(\frac{s - x\cdot e}{c}, x\right)$$

for all $s \in \mathbb{R}$ and $x \in \mathbb{R}^N$, where u is a classical solution of (1.10-1.11).

The function ϕ satisfies the following degenerate elliptic equation

$$\begin{aligned} \nabla_x\cdot(A(x)\nabla_x\phi) + (eA(x)e)\phi_{ss} + \nabla_x\cdot(A(x)e\phi_s) \\ +\partial_s(eA(x)\nabla_x\phi) - c\phi_s + f(x,\phi) \; = 0 \quad \text{in } \mathcal{D}'_L(\mathbb{R}\times\mathbb{R}^N) \end{aligned} \tag{2.13}$$

together with the periodicity condition

$$\phi \text{ is L-periodic with respect to } x. \tag{2.14}$$

Moreover, since $u(t,x) \to 0$ as $x\cdot e \to -\infty$ and $u(t,x) - p(x) \to 0$ as $x\cdot e \to +\infty$, locally in t and uniformly in the directions of $\mathbb{R}^N$ which are orthogonal to e, and since ϕ is L-periodic with respect to x, one gets

$$\phi(-\infty,x) = 0, \quad \phi(+\infty,x) = p(x) \text{ uniformly in } x \in \mathbb{R}^N. \tag{2.15}$$

Conversely, if ϕ is a solution of (2.13-2.15) such that $u(t,x) = \phi(x\cdot e + ct, x)$ is C^1 in t, C^2 in x, then u is a classical solution of (1.10-1.11).

Let a and ε be two positive real numbers, and set $\Sigma_a = (-a,a)\times\mathbb{R}^N$. As it was done in [4], one first works with elliptic regularizations of (2.13) of the type

$$\left\{ \begin{array}{rcl} L_\varepsilon\phi + f(x,\phi) & = & 0 \text{ in } \Sigma_a, \\ \phi \text{ is L-periodic w.r.t. } x, & & \\ \forall x \in \mathbb{R}^N, \quad \phi(-a,.) = 0,\ \phi(a,x) & = & p(x), \end{array} \right. \tag{2.16}$$

where L_ε is the elliptic (in the (s,x)-variables) operator defined by

$$\begin{aligned} L_\varepsilon\phi \;=\; & \nabla_x\cdot(A(x)\nabla_x\phi) + (eA(x)e + \varepsilon)\phi_{ss} \\ & +\nabla_x\cdot(A(x)e\phi_s) + \partial_s(eA(x)\nabla_x\phi) - c\phi_s. \end{aligned}$$

We will follow the scheme as in [4] to prove the existence of solutions of (2.16) and state some of their properties, only indicating the differences which may appear.

Let us establish at first the

Lemma 2.1 *For each $c \in \mathbb{R}$, there exists a solution $\phi^c \in C^2\left(\overline{\Sigma_a}\right)$ of (2.16).*

PROOF. Let ψ be the function defined by $\psi(s,x) = p(x)\dfrac{s+a}{2a}$. One sets $v = \phi - \psi$. Then, since p satisfies $-\nabla\cdot(A(x)\nabla p) - f(x,p) = 0$ in $\mathbb{R}^N$, the problem (2.16) is equivalent to

$$\left\{\begin{array}{rcl} -L_\varepsilon v & = & f(x, v+\psi) - \dfrac{s+a}{2a} f(x,p) \\ & & +(2a)^{-1}[2A(x)e\cdot\nabla p + \nabla\cdot(A(x)e)\ p - cp] \ \text{ in } \Sigma_a, \\ & & v \text{ is L-periodic w.r.t. } x, \\ v(-a,.) & = & 0, \quad v(a,.) \ = \ 0. \end{array}\right. \tag{2.17}$$

Using the fact that $f(x,p)$, p, A, ∇p and $\nabla_x\cdot(Ae)$ are globally bounded, (since p and A are L-periodic and C^1), one can follow the proof of Lemma 5.1 of [4]. Namely, using Lax-Milgram Theorem with Schauder fixed point Theorem, one can find a solution v of the first equation of (2.17), in the distribution sense, in Σ_a. Then, from the regularity theory up to the boundary, this solution v is a classical solution of (2.17) in $\overline{\Sigma_a}$. Finally, the function $\phi = v + \psi \in C^2\left(\overline{\Sigma_a}\right)$ is a classical solution of (2.16). □

Lemma 2.2 *The function ϕ^c defined above is increasing in s and it is the unique solution of (2.16) in $C^2\left(\overline{\Sigma_a}\right)$.*

PROOF. One has to show at first that $0 < \phi^c(s,x) < p(x)$ in Σ_a. Since $f \equiv 0$ in $\mathbb{R}^N \times (-\infty, 0]$, the strong elliptic maximum principle yields that $\phi^c > 0$ in $(-a,a)\times\mathbb{R}^N$. Let us show that $\phi^c(s,x) < p(x)$.

Set

$$\gamma^* = \inf\left\{\gamma,\ \gamma p(x) > \phi^c(s,x) \text{ for all } (s,x)\ \in \overline{\Sigma_a}\right\}.$$

Since $p > 0$ and p is L-periodic with respect to x, there exists $\delta > 0$ such that $p > \delta$ in $\overline{\Sigma_a}$. Therefore γ^* does exist. Moreover since $\phi^c(a,x) = p(x)$, $\gamma^* \geq 1$. One has to show that $\gamma^* = 1$.

Assume $\gamma^* > 1$. By continuity, one has $\gamma^* p \geq \phi^c$ in $\overline{\Sigma_a}$. On the other hand, there exists a sequence $\gamma_n \to \gamma^*$, $\gamma_n < \gamma^*$ and a sequence (s_n, x_n) in $\overline{\Sigma_a}$ such that $\gamma_n p(x_n) \leq \phi^c(s_n, x_n)$. Since p and ϕ^c are L-periodic in x, one can assume that $x_n \in \overline{C}$. Up to the extraction of a subsequence, one can also assume that $(s_n, x_n) \to (s_1, x_1) \in [-a,a]\times\overline{C}$. Passing to the limit $n\to\infty$, one obtains $\gamma^* p(x_1) = \phi^c(s_1, x_1)$.

Next, set $z = \gamma^* p - \phi^c$. One has $f(x, \gamma^* p) \leq \gamma^* f(x,p)$ since $f(.,s)/s$ is supposed to be decreasing in s. Thus $L_\varepsilon(\gamma^* p) + f(x,\gamma^* p) \leq 0$. As a consequence, $L_\varepsilon z + f(x,\gamma^* p) - f(x,\phi^c) \leq 0$.

Hence,

$$\left\{\begin{array}{l} L_\varepsilon z + bz \leq 0 \\ z \geq 0 \end{array}\right. \text{ in } \Sigma_a, \tag{2.18}$$

where b is a bounded function (because f is globally lipschitz-continuous). Moreover, one has

$$\left\{\begin{array}{rcl} \gamma^* p(x) & > & \phi^c(-a,x) = 0, \\ \gamma^* p(x) & > & \phi^c(a,x) = p(x), \end{array}\right. \tag{2.19}$$

for all x in $\mathbb{R}^N$ (the last inequality follows from the assumption $\gamma^* > 1$ and from the positivity of p in $\mathbb{R}^N$). Therefore, the point (s_1, x_1) where z vanishes, lies in $(-a, a) \times \overline{C}$. Using (2.18) with the strong maximum principle, one obtains that $z \equiv 0$, which contradicts (2.19).

Thus $\gamma^* = 1$ and $\phi^c \leq p$. Using again the strong maximum principle, one obtains $\phi^c(s, x) < p(x)$ for all $(s, x) \in [-a, a) \times \mathbb{R}^N$.

In order to finish the proof of the lemma, we only have to follow the proof of Lemma 5.2 given in [4], which uses a sliding method in s (see [11]), replacing $\phi^c(a, x) = 1$ by $\phi^c(a, x) = p(x)$. □

Lemma 2.3 *The functions ϕ^c are decreasing and continuous with respect to c in the following sense : if $c > c'$, then $\phi^c < \phi^{c'}$ in Σ_a and if $c_n \to c \in \mathbb{R}$, then $\phi^{c_n} \to \phi^c$ in $C^2(\overline{\Sigma_a})$.*

PROOF. The proof is similar to that of lemma 5.3 in [4]. □

In the following, for any $\varepsilon > 0$, $a > 0$ and $c \in \mathbb{R}$, we call $\phi^c_{\varepsilon,a}$ the unique solution of (2.16) in $C^2\left(\overline{\Sigma_a}\right)$.

Set

$$p^- = \min_{x \in \overline{C}} \; p(x) = \min_{x \in \mathbb{R}^N} \; p(x) \; > 0 \text{ and } p^+ = \max_{x \in \overline{C}} = \max_{x \in \mathbb{R}^N} \; p(x) \; > 0.$$

Lemma 2.4 *There exist a_1 and K such that, for all $a \geq a_1$ and $\varepsilon \in (0, 1]$,*

$$(c > K) \Rightarrow \left(\max_{x \in \overline{C}} \phi^c_{\varepsilon,a}(0, x) < \frac{p^-}{2} \right).$$

PROOF. Let $n \geq 2$ be an integer and g be a C^1 function defined on $[0, np^+]$, such that $g(0) = 0$, $g(np^+) = 0$, $g(u) > 0$ on $(0, np^+)$, $g'(np^+) < 0$. For n large enough, one can choose (using hypothesis (1.8)) g such that $f(x, u) \leq g(u)$ for all $x \in \mathbb{R}^N$ and $u \in [0, np^+]$.

Then, using a result of [9], one can assert that there exists c^1 such that the one-dimensional problem

$$\begin{cases} v'' - kv' + \dfrac{g(v)}{\alpha_0} = 0 \text{ in } \mathbb{R}, \\ v(-\infty) = 0 < v(.) < v(+\infty) = np^+ \text{ and } v(0) = \dfrac{p^-}{2}, \end{cases} \tag{2.20}$$

admits a unique solution v, for each $k \geq c^1 > 0$ (remember that $\alpha_0 > 0$ is given in (1.6)). Set $k = c^1$, and let $v = v(s)$ be the unique solution of (2.20) associated to $k = c^1$. It is also known that v is increasing in $\mathbb{R}$. Take $c > \max_{x \in \mathbb{R}^N} \{(eA(x)e + 1)k + \nabla \cdot (A(x)e)\}$, and $\varepsilon \in (0, 1]$. One has

$$\begin{aligned} L_\varepsilon v + f(x, v) &= (eA(x)e + \varepsilon)v''(s) + (\nabla \cdot (A(x)e) - c)v'(s) + f(x, v(s)) \\ &= \{(eA(x)e + \varepsilon)k + \nabla \cdot (A(x)e) - c\} \, v'(s) \\ &\quad -(eA(x)e + \varepsilon)g(v(s))/\alpha_0 + f(x, v(s)), \end{aligned}$$

from (2.20). Thus

$$L_\varepsilon v + f(x,v) \leq \{(eA(x)e+\varepsilon)k + \nabla(\cdot A(x)e) - c\}\, v'(s) - \varepsilon g(v(s))/\alpha_0,$$

since $eA(x)e \geq \alpha_0$ and $f(x,u) \leq g(u)$ for all $(x,u) \in \mathbb{R}^N \times [0, np^+]$. Moreover $g \geq 0$, $v' > 0$ and $(eA(x)e+\varepsilon)k + \nabla \cdot A(x)e - c < 0$, owing to the choice of c. As a consequence, one gets

$$L_\varepsilon v + f(x,v) < 0 \text{ in } \Sigma_a.$$

By using a sliding method as in Lemma 2.2, with v and $\phi^c_{\varepsilon,a}$, and by using the monotonicity of v and the fact that $v(-a) > 0$ and $v(a) > p^+ \geq p(x)$ for all $x \in \mathbb{R}^N$ and for a large enough, one can conclude that

$$\phi^c_{\varepsilon,a}(s,x) < v(s),$$

for all $(s,x) \in \overline{\Sigma_a}$ and for a large enough.

Hence, it follows that

$$\max_{x\in\mathbb{R}^N} \phi^c_{\varepsilon,a}(0,x) = \max_{x\in\overline{C}} \phi^c_{\varepsilon,a}(0,x) < v(0) = \frac{p^-}{2},$$

for $c > \max\limits_{\mathbb{R}^N} \{(eA(x)e+1)k + \nabla\cdot(A(x)e)\}$ and a large enough, which completes the proof of the lemma. □

Let us now consider the functions $\phi^0_{\varepsilon,a}$, associated to $c=0$. Take a sequence $a_n \to +\infty$. Let us pass to the limit $n \to +\infty$. From standard elliptic estimates and Sobolev's injections, the functions ϕ^0_{ε,a_n} converge (up to the extraction of a subsequence) in $C^{2,\beta}_{loc}(\mathbb{R}\times\mathbb{R}^N)$, for all $0 \leq \beta < 1$, to a function ϕ^0 which satisfies

$$\begin{cases} L_\varepsilon \phi^0 + f(x,\phi^0) = 0 \text{ in } \mathbb{R}\times\mathbb{R}^N, \\ \phi^0 \text{ is L-periodic w.r.t. } x, \\ \phi^0 \text{ is nondecreasing w.r.t. } s, \end{cases}$$

with $c=0$. Furthermore, $0 \leq \phi^0(s,x) \leq p(x)$ for all $(s,x) \in \mathbb{R}\times\mathbb{R}^N$.

One then has the

Lemma 2.5 *There exist $x_1 \in \overline{C}$ and $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$, $\phi^0_{\varepsilon,a_n}(0,x_1) > \dfrac{p^-}{2}$.*

PROOF. Assume $\phi^0(0,x) < p(x)$ for all $x \in \overline{C}$. Then

$$0 \leq \max_{x\in\overline{C}} \phi^0(0,x) < p^+.$$

Let $(s_n)_{n\in\mathbb{N}}$ be a sequence in $(-a,a)$, such that

$$\max_{x\in\overline{C}} \phi^0_{\varepsilon,a_n}(s_n,x) = b := \frac{1}{2}p^+ + \frac{1}{2}\max_{x\in\overline{C}} \phi^0(0,x).$$

Let us show that $s_n > 0$ for n large enough.

Assume by contradiction that there exists a subsequence $a_k \to +\infty$ such that $s_k \le 0$ for all $k \in \mathbb{N}$. Then, since ϕ^0_{ε,a_k} is increasing in s,

$$b = \max_{x\in\overline{C}} \phi^0_{\varepsilon,a_k}(s_k, x) \le \max_{x\in\overline{C}} \phi^0_{\varepsilon,a_k}(0, x),$$

while

$$\max_{x\in\overline{C}} \phi^0_{\varepsilon,a_k}(0, x) \to \max_{x\in\overline{C}} \phi^0(0, x) \text{ as } k \to +\infty.$$

Passing to the limit $k \to +\infty$, one obtains

$$b \le \max_{x\in\overline{C}} \phi^0(0, x).$$

That leads to a contradiction since $p^+ > \max_{x\in\overline{C}} \phi^0(0, x)$. Therefore, one has shown that $s_n > 0$ for n large enough.

Set $\phi_{a_n}(s, x) = \phi^0_{\varepsilon,a_n}(s + s_n, x)$, defined on $(-a_n - s_n, a_n - s_n) \times \mathbb{R}^N$. Then

$$\max_{x\in\overline{C}} \phi_{a_n}(0, x) = b.$$

One easily sees that $-a_n - s_n \to -\infty$ as $n \to +\infty$. Then two cases may occur, up to the extraction of some subsequence :

case 1 : $a_n - s_n \to +\infty$. From standard elliptic estimates and Sobolev's injections, the functions ϕ_{a_n} converge (up to the extraction of a subsequence) in $C^{2,\beta}_{loc}(\mathbb{R} \times \mathbb{R}^N)$, for all $0 \le \beta < 1$, to a nonnegative function ϕ satisfying

$$\begin{cases} L_\varepsilon \phi + f(x, \phi) = 0 \text{ in } \mathbb{R} \times \mathbb{R}^N \text{ (with } c = 0), \\ \phi \text{ is L-periodic w.r.t. } x, \\ \phi \text{ is nondecreasing w.r.t. } s, \\ 0 \le \phi \le p(x) \\ \max_{x\in\overline{C}} \phi(0, x) = b. \end{cases} \tag{2.21}$$

Moreover, from standard elliptic estimates, from the monotonicity of ϕ, and from the periodicity in x, it follows that

$$\phi(s, x) \to \phi_\pm(x) \text{ in } C^2(\mathbb{R}^N) \text{ as } s \to \pm\infty,$$

where each function $\phi_\pm$ satisfies

$$\begin{cases} \nabla \cdot (A(x)\nabla\phi_\pm) + f(x, \phi_\pm) = 0 \text{ in } \mathbb{R}^N, \\ \phi_\pm \text{ is L-periodic}, \\ \phi_\pm \ge 0. \end{cases}$$

¿From the uniqueness Theorem 2.1 of [8], one can conclude that either $\phi_\pm(x) \equiv 0$ or $\phi_\pm(x) \equiv p(x)$. Moreover, $\phi_1^-(x) \le \phi_1(0, x)$ because of the monotonicity of ϕ_1 with respect to s. Therefore $\phi_1^-(x) \le b < p^+$. Thus, $\phi_1^-(x) \not\equiv p(x)$, and $\phi_1^- \equiv 0$. Similarly, $\phi_1^+(x) \ge \phi_1(0,x)$. Thus there exists $x_0 \in \overline{C}$ such that $\phi_1^+(x_0) \ge b > 0$. As a consequence, $\phi_1^+ \equiv p$.

Next, multiply the equation (2.21) (with $c = 0$) by ϕ_s and integrate it over $(-N,N)\times C$, where N is a positive real number. Then,

$$\begin{aligned}\int_{(-N,N)\times C} (eA(x)e+\varepsilon)\phi_{ss}\phi_s dsdx + \int_{(-N,N)\times C} \nabla_x\cdot(A(x)\nabla_x\phi)\phi_s dsdx \\ +\int_{(-N,N)\times C} \{\nabla_x\cdot(A(x)e\phi_s)+\partial_s(eA(x)\nabla_x\phi)\}\,\phi_s dsdx \\ +\int_{(-N,N)\times C} f(x,\phi)\phi_s dsdx \;=\; 0.\end{aligned} \tag{2.22}$$

First, one has

$$\int_{(-N,N)\times C} (eA(x)e+\varepsilon)\phi_{ss}\phi_s dsdx = \frac{1}{2}\int_C \left[(eA(x)e+\varepsilon)(\phi_s)^2\right]_{-N}^N dsdx. \tag{2.23}$$

¿From standard elliptic estimates, one knows that $\phi_s \to 0$ as $s \to +\infty$. Passing to the limit $N \to +\infty$ in (2.23), one obtains

$$\int_{\mathbb{R}\times C} (eA(x)e+\varepsilon)\phi_{ss}\phi_s dsdx = 0. \tag{2.24}$$

Next, using an integration by parts over $(-N,N)\times C$ and the periodicity of ϕ with respect to x, one has

$$\begin{aligned}\int_{(-N,N)\times C} \nabla_x\cdot(A(x)\nabla_x\phi)\phi_s dsdx &= -\int_{(-N,N)\times C} \nabla_x\phi_s\cdot A(x)\nabla_x\phi dsdx \\ &= -\frac{1}{2}\int_{(-N,N)\times C} (\nabla_x\phi\cdot A(x)\nabla_x\phi)_s dsdx \\ &= -\frac{1}{2}\int_C \left[\nabla_x\phi\cdot A(x)\nabla_x\phi\right]_{-N}^N dsdx,\end{aligned} \tag{2.25}$$

since the matrix field $A(x)$ is symmetric. Passing to the limit $N \to +\infty$ in (2.25), one obtains, using standard elliptic estimates :

$$\int_{\mathbb{R}\times C} \nabla_x\cdot(A(x)\nabla_x\phi)\phi_s dsdx = -\frac{1}{2}\int_C \nabla p\cdot(A(x)\nabla p)dsdx. \tag{2.26}$$

Next, from the periodicity of ϕ with respect to x, one can similarly show that

$$\int_{\mathbb{R}\times C} \{\nabla_x\cdot(A(x)e\phi_s)+\partial_s(eA(x)\nabla_x\phi)\}\,\phi_s dsdx = 0. \tag{2.27}$$

Set $F(x,u) = \displaystyle\int_0^u f(x,s)ds$. Then,

$$\int_{\mathbb{R}\times C} f(x,\phi)\phi_s dsdx = \int_{\mathbb{R}\times C} F(x,\phi(s,x))_s dsdx = \int_C F(x,p(x))dx. \tag{2.28}$$

Passing to the limit $N \to +\infty$ in (2.22), and using (2.24), (2.26), (2.27) and (2.28), one gets

$$\int_C \left[F(x,p(x)) - \frac{1}{2}\nabla p \cdot (A(x)\nabla p)\right] dx = 0. \tag{2.29}$$

Moreover, using a property on the energy of p, which has been established in Proposition 3.7 of [8], one asserts that

$$\int_C \left[F(x,p(x)) - \frac{1}{2}\nabla p \cdot (A(x)\nabla p)\right] dx =: -E(p) > 0.$$

The latter is in contradiction with (2.29), therefore case 1 is ruled out.

case 2 : $a_n - s_n \to b < +\infty$. Up to the extraction of some subsequence, the functions ϕ_{a_n} converge in $C^{2,\beta}_{loc}((-\infty, b) \times \mathbb{R}^N)$ (for all $0 \le \beta < 1$) to a function ϕ satisfying (2.21), with $c = 0$, in the set $(-\infty, b) \times \mathbb{R}^N$. Moreover, the family of functions (ϕ_{a_n}) is equi-Lipschitz-continuous in any set of the type $[a_n - s_n - 1, a_n - s_n] \times \overline{C}$. Therefore, for all $\eta > 0$, there exists $\kappa > 0$ such that

$$\forall x \in \overline{C},\ \forall n,\ \forall s \in [a_n - s_n - \kappa, a_n - s_n],\ p(x) - \eta \le \phi_{a_n}(s,x) \le p(x). \tag{2.30}$$

Then choose $x_0 \in \overline{C}$ such that $p(x_0) = p^+$. Formula (2.30) applied on x_0 together with

$$\max_{x \in \overline{C}} \phi_{a_n}(0,x) = b < p^+$$

implies that $a_n - s_n > \delta$ for some $\delta > 0$. Hence $b > 0$ and

$$\max_{x \in \overline{C}} \phi(0,x) = b.$$

Moreover (2.30) implies that ϕ can be extended by continuity on $\{b\} \times \mathbb{R}^N$ with $\phi(b,x) = p(x)$. Furthermore, from standard elliptic estimates up to the boundary, the function ϕ is actually in $C^1\left((-\infty, b] \times \mathbb{R}^N\right)$.

Following the proof of case 1, one shows that $\phi(-\infty, .) \equiv 0$. The next steps are similar to those of case 1. One gets a contradiction.

Therefore, there exists $x_1 \in \overline{C}$ such that $\phi^0(0,x_1) = p(x_1)$. Hence, taking any sequence $a_n \to +\infty$ such that the sequence $(\phi^0_{\varepsilon,a_n})$ converges in $C^{2,\beta}_{loc}(\mathbb{R} \times \mathbb{R}^N)$ for all $0 \le \beta < 1$, there exists N_0 such that for all $n \ge N_0$, $\phi^0_{\varepsilon,a_n}(0,x_1) > \dfrac{p^-}{2}$. That completes the proof of Lemma 2.5. □

Finally, one gets

Proposition 2.6 *Fix $\varepsilon \in (0,1]$. Let $a_n \to +\infty$ be the sequence defined above. Then, there exist $K \in \mathbb{R}$, $N_1 \in \mathbb{N}$ such that for all $n \ge N_1$ there exists a unique real number $c = c^{\varepsilon,a_n}$ such that ϕ^c_{ε,a_n} satisfies the normalization condition*

$$\max_{x \in \mathbb{R}^N} \phi^c_{\varepsilon,a_n}(0,x) = \max_{x \in \overline{C}} \phi^c_{\varepsilon,a_n}(0,x) = \frac{p^-}{2}. \tag{2.31}$$

Furthermore,

$$\forall\ 0 < \varepsilon \leq 1,\ \forall\ n \geq N_1,\ 0 < c^{\varepsilon,a_n} < K.$$

PROOF. Fix $\varepsilon \in (0,1]$. Under the notations of the two preceding lemmas, let us define N_1 such that $a_{N_1} > a_1$ and $N_1 > N_0$. It follows from this lemmas that for each $n \geq N_1$,

$$\begin{cases} \forall c \geq K,\ \max_{x\in\mathbb{R}^N} \phi^c_{\varepsilon,a_n}(0,x) < \dfrac{p^-}{2}, \\ \text{for } c = 0,\ \max_{x\in\mathbb{R}^N} \phi^0_{\varepsilon,a_n}(0,x) > \dfrac{p^-}{2}. \end{cases}$$

On the other hand, Lemma 2.3 yields that, for each $n \geq N_1$, the function

$$\Xi(c) = \max_{x\in\mathbb{R}} \phi^c_{\varepsilon,a_n}(0,x)$$

is decreasing and continuous with respect to c. Therefore the proposition follows. □

2.2 Passage to the limit in the unbounded domains

Using the result of Proposition 2.6, we are going to pass to the limit $n \to +\infty$ in the unbounded domain $\mathbb{R}\times\mathbb{R}^N$ for the solutions $\phi^{c^{\varepsilon,a_n}}_{\varepsilon,a_n}$ satisfying (2.31).

Proposition 2.7 *Under the notations of Proposition 2.6, one has*

$$\forall \varepsilon > 0,\ 0 < c^\varepsilon := \liminf_{n\to+\infty, n\geq N_1} c^{\varepsilon,a_n} \leq K.$$

PROOF. From Proposition 2.6, one has $0 \leq c^\varepsilon \leq K$. Up to the extraction of a subsequence, one can assume $c^{\varepsilon,a_n} \to c^\varepsilon$ as $n \to +\infty$ and $\phi^{c^{\varepsilon,a_n}}_{\varepsilon,a_n} \to \phi$ in $C^{2,\beta}_{loc}(\mathbb{R}\times\mathbb{R}^N)$, for all $0 \leq \beta < 1$, where ϕ satisfies

$$\begin{cases} L_\varepsilon\phi + f(x,\phi) = 0 \text{ in } \mathbb{R}\times\mathbb{R}^N, \\ \phi \text{ is L-periodic w.r.t. } x, \\ \phi \text{ is nondecreasing w.r.t. } s, \end{cases}$$

with $c = c^\varepsilon$ and

$$\max_{x\in\mathbb{R}^N} \phi(0,x) = \frac{p^-}{2}.$$

Then, following the calculus of Lemma 2.5, case 1, one can assert that $\phi(-\infty,x) = 0$, $\phi(+\infty,x) = p(x)$ for all $x \in \mathbb{R}$ and

$$c^\varepsilon \int_{\mathbb{R}\times C} (\phi_s)^2 ds dx = \int_C \left[F(x,p) - \frac{1}{2}\nabla p \cdot (A(x)\nabla p)\right] dx = -E(p) > 0, \tag{2.32}$$

from Proposition 3.7 of [8]. Therefore $c^\varepsilon > 0$. □

Proposition 2.8 *Up to the extraction of some subsequence, the functions $\phi_{\varepsilon,a_n}^{c^{\varepsilon,a_n}}$ converge in $C^{2,\beta}_{loc}(\mathbb{R}\times\mathbb{R}^N)$ (for all $0\le\beta<1$), to a function ϕ^ε such that, in $\mathbb{R}\times\mathbb{R}^N$,*

$$\begin{cases} \nabla_x\cdot(A(x)\nabla_x\phi^\varepsilon)+(eA(x)e+\varepsilon)\phi^\varepsilon_{ss}+\nabla_x\cdot(A(x)e\phi^\varepsilon_s) & \\ \qquad +\partial_s(eA(x)\nabla_x\phi^\varepsilon)-c\phi^\varepsilon_s+f(x,\phi^\varepsilon) & =0, \\ \phi^\varepsilon \text{ is } L\text{-periodic w.r.t. } x, & \\ \max_{x\in\mathbb{R}^N}\phi^\varepsilon(0,x) & =\dfrac{p^-}{2}, \\ \phi^\varepsilon \text{ is increasing w.r.t. } s. & \end{cases}$$

Furthermore, $\phi^\varepsilon(-\infty,x)=0$ and $\phi^\varepsilon(+\infty,x)=p(x)$ for all $x\in\mathbb{R}^N$.

PROOF. The convergence follows from the same arguments that were used in the preceding propositions. Moreover, ϕ^ε in nondecreasing w.r.t. s because each $\phi_{\varepsilon,a_n}^{c^{\varepsilon,a_n}}$ is increasing in s. The limits $\phi^\varepsilon(-\infty,x)=0$ and $\phi^\varepsilon(+\infty,x)=p(x)$ can be proved in the same way as in Lemma 2.5, case 1, using

$$\max_{x\in\mathbb{R}^N}\phi^\varepsilon(0,x)=\frac{p^-}{2}$$

and the fact that ϕ^ε in nondecreasing in s. The only thing that it remains to prove is that ϕ^ε is increasing in s.

For any $h>0$, the function $\phi^\varepsilon(s+h,x)-\phi^\varepsilon(s,x)$ is a nonnegative and nonconstant solution of a linear elliptic equation with bounded coefficients. It follows then from the strong maximum principle that $\phi^\varepsilon(s+h,x)-\phi^\varepsilon(s,x)>0$, for all $(s,x)\in\mathbb{R}\times\mathbb{R}^N$. That proves that the function ϕ^ε is increasing in the variable s. □

2.3 Passage to the limit $\varepsilon\to 0$

Our first aim is to prove that the real numbers c^ε are bounded from below by a positive constant.

Proposition 2.9 *Under the notations of Proposition 2.6, one has*

$$0<\liminf_{\varepsilon\to 0}c^\varepsilon\le K.$$

PROOF. From Proposition 2.6, for each $\varepsilon>0$, one has $0<c^\varepsilon\le K$. Assume that there exists a sequence $\varepsilon_n\to 0$, $\varepsilon_n>0$ such that $c^{\varepsilon_n}\to 0$ as $n\to+\infty$. In the sequel, for the sake of simplicity, we drop the index n. Set $u^\varepsilon(t,x)=\phi^\varepsilon(x\cdot e+c^\varepsilon t,x)$. Then u^ε is a classical solution of

$$\begin{cases} \dfrac{\varepsilon}{(c^\varepsilon)^2}u^\varepsilon_{tt}+\nabla_x\cdot(A(x)\nabla_x u^\varepsilon)-u^\varepsilon_t+f(x,u^\varepsilon)=0 \text{ in } \mathbb{R}\times\mathbb{R}^N, \\ \forall\ k\in\displaystyle\prod_{i=1}^N L_i\mathbb{Z},\ u^\varepsilon\left(t+\frac{k\cdot e}{c},x\right)=u^\varepsilon(t,x+k) \quad \text{in } \mathbb{R}\times\mathbb{R}^N, \\ u^\varepsilon(t,x)\to 0 \text{ as } t\to-\infty,\ u^\varepsilon(t,x)\to p(x) \text{ as } t\to+\infty. \end{cases} \tag{2.33}$$

Moreover, $0 < u^\varepsilon(t,x) < p(x)$ for all $(t,x) \in \mathbb{R}\times\mathbb{R}^N$. Lastly, since ϕ is increasing in the variable s and $c^\varepsilon > 0$, each function u^ε is increasing in the variable t.

Up to the extraction of some subsequence, as it was said in [4] (Proposition 5.10), three cases may occur :

$$\frac{\varepsilon}{(c^\varepsilon)^2} \to \kappa \in (0,+\infty),\ \frac{\varepsilon}{(c^\varepsilon)^2} \to +\infty \text{ or } \frac{\varepsilon}{(c^\varepsilon)^2} \to 0 \ \text{ as } \varepsilon \to 0^+.$$

Let us study the :

case 1 : Assume $\dfrac{\varepsilon}{(c^\varepsilon)^2} \to \kappa \in (0,+\infty)$. Let $x_0\ \mathbb{R}^N$ be such that $x_0 \in \prod_{i=1}^N L_i\mathbb{Z}$ and $x_0\cdot e > 0$. Since $u^\varepsilon(t,x_0) \to 0$ as $t \to -\infty$ and $u^\varepsilon(t,x_0) \to p(x_0)$ as $t \to +\infty$ from our assumptions on u^ε, one can assume, up to translation with respect to t, that $u^\varepsilon(0,x_0) = \dfrac{p^-}{2}$.

Since $\dfrac{\varepsilon}{(c^\varepsilon)^2} \to \kappa$, standard elliptic estimates imply that the functions u^ε converge (up to extraction of some subsequence) in $C^{2,\beta}_{loc}(\mathbb{R}\times\mathbb{R}^N)$ (for all $0 \le \beta < 1$) to a function u satisfying

$$\begin{cases} \kappa u_{tt} + \nabla_x\cdot(A(x)\nabla_x u) - u_t + f(x,u) &= 0 \text{ in } \mathbb{R}\times\mathbb{R}^N, \\ 0 \le u \le p,\ u_t &\ge 0 \text{ in } \mathbb{R}\times\mathbb{R}^N, \end{cases}$$

and $u(0,x_0) = \dfrac{p^-}{2}$. Now, fix any $B \in \mathbb{R}$. Since $c^\varepsilon \to 0^+$ and $x_0\cdot e > 0$, $B < \frac{x_0\cdot e}{c^\varepsilon}$ for ε sufficiently small. Thus, as u^ε is increasing in t, one has $u^\varepsilon(B,0) \le u^\varepsilon(\frac{x_0\cdot e}{c^\varepsilon},0)$. But $u^\varepsilon(\frac{x_0\cdot e}{c^\varepsilon},0) = u^\varepsilon(0,x_0) = \frac{p^-}{2}$. Therefore, passing to the limit $\varepsilon \to 0$, one obtains

$$\forall\ B > 0, \qquad u(B,0) \le \frac{p^-}{2}. \tag{2.34}$$

Let u^+ be the function defined in $\mathbb{R}^N$ by $u^+(x) = \lim_{t\to+\infty} u(t,x)$. This function can be defined since u is bounded and nondecreasing in t. From standard elliptic estimates, the convergence holds in $C^2_{loc}(\mathbb{R}^N)$, and u^+ solves

$$\nabla\cdot(A(x)\nabla u^+) + f(x,u^+) = 0 \text{ in } \mathbb{R}^N \tag{2.35}$$

with $0 \le u^+(x) \le p(x)$ for all $x \in \mathbb{R}^N$. But it follows from our hypotheses on f and from Theorems 2.1 and 2.3 of [8] that the equation (2.35) admits exactly two nonnegative solutions, which are 0 and p. Therefore, as $u(0,x_0) = \frac{p^-}{2}$ and $u_t \ge 0$, one has $u^+(x_0) \ge \frac{p^-}{2} > 0$, thus $u^+ \equiv p$. However, (2.34) gives $u^+(0) \le \frac{p^-}{2}$. As a consequence, u^+ cannot be equal to p and case 1 is ruled out.

case 2 : Assume that $\dfrac{\varepsilon}{(c^\varepsilon)^2} \to +\infty$. As it was done in [4] (Proposition 5.10), one makes the change of variables $\tau = (c^\varepsilon/\sqrt{\varepsilon})t$. The function $v^\varepsilon(\tau,x) = u^\varepsilon\left(\dfrac{\sqrt{\varepsilon}}{c^\varepsilon}\tau, x\right)$

satisfies

$$\begin{cases} v^\varepsilon_{\tau\tau} + \nabla_x \cdot (A(x)\nabla_x v^\varepsilon) - \dfrac{c^\varepsilon}{\sqrt{\varepsilon}} v^\varepsilon_\tau + f(x, v^\varepsilon) = & 0 \text{ in } \mathbb{R} \times \mathbb{R}^N, \\ \forall\ k \in \displaystyle\prod_{i=1}^{N} L_i\mathbb{Z}\ v^\varepsilon\left(\tau + \dfrac{k \cdot e}{\sqrt{\varepsilon}}, x\right) = & v^\varepsilon(\tau, x+k) \ \text{ in } \mathbb{R} \times \mathbb{R}^N, \\ v^\varepsilon(\tau, x) \to 0 \text{ as } \tau \to -\infty,\ v^\varepsilon(\tau, x) - p(x) \to 0 \ \text{ as } \tau \to +\infty. \end{cases}$$

Moreover, $0 < v^\varepsilon < p$ and v^ε is nondecreasing with respect to τ. Furthermore, as it was done in case 1, one can assume that $v^\varepsilon(0, x_0) = \frac{p^-}{2}$ for some $x_0 \in \mathbb{R}^N$ such that $x_0 \in \prod_{i=1}^N L_i\mathbb{Z}$ and $x_0 \cdot e > 0$. Since $c^\varepsilon/\sqrt{\varepsilon} \to 0^+$, the functions v^ε converge (up to the extraction of some subsequence) in $C^{2,\beta}_{loc}(\mathbb{R} \times \mathbb{R}^N)$ (for all $0 \le \beta < 1$) to a function v which satisfies

$$\begin{cases} v_{\tau\tau} + \nabla_x \cdot (A(x)\nabla_x v) + f(x, v) &= 0 \text{ in } \mathbb{R} \times \mathbb{R}^N, \\ 0 \le v \le p,\ v_\tau &\ge 0 \text{ in } \mathbb{R} \times \mathbb{R}^N, \end{cases}$$

and $v(0, x_0) = \frac{p^-}{2}$. Moreover, as in case 1, one can show that

$$\forall\ B > 0, \quad v(B, 0) \le \frac{p^-}{2}.$$

By defining $v^+(x) := \lim_{\tau \to +\infty} v(\tau, x)$, one can also obtain a contradiction the same way as in case 1.

case 3 : Assume that $\dfrac{\varepsilon}{(c^\varepsilon)^2} \to 0$. The elliptic operators in (2.33) become degenerate at the limit $\varepsilon \to 0$, and one cannot use the same arguments as in cases 1 and 2.

In order to pass to the limit $\varepsilon \to 0$, let us state two new inequalities on u^ε.

Using the same calculations as those which were used to prove (2.29) and (2.32), and making the change of variables $t = \dfrac{s - x \cdot e}{c^\varepsilon}$, one obtains

$$\forall\ \varepsilon \in (0,1), \int_{\mathbb{R} \times C} (u^\varepsilon_t)^2 = -E(p).$$

Therefore, the periodicity condition in (2.33) gives us that

$$\forall \varepsilon \in (0,1),\ \forall n \in \mathbb{N}, \quad \int_{\mathbb{R} \times (-nL_1, nL_1) \times \cdots \times (-nL_N, nL_N)} (u^\varepsilon_t)^2 = -(2n)^N E(p). \tag{2.36}$$

Similarly, multiplying equation (2.33) by 1 and u^ε, one gets the existence of $\gamma \ge 0$ such that

$$\forall \varepsilon \in (0,1),\ \forall n \in \mathbb{N}, \quad \int_{\mathbb{R} \times (-nL_1, nL_1) \times \cdots \times (-nL_N, nL_N)} f(x, u^\varepsilon) + |\nabla_x u^\varepsilon|^2 \le (2n)^N \gamma. \tag{2.37}$$

Next, using Theorem A.1 of [4] (see also [5]), one has the following a priori estimate :

Lemma 2.10 *There exists a constant M, which does not depend on ε, such that the function u^ε solving (2.33) satisfies*

$$|\nabla_x u^\varepsilon| \leq M \ \text{in}\ \mathbb{R}\times\mathbb{R}^N \tag{2.38}$$

for ε small enough.

The above estimates were proved in [5] with a Bernstein-type method. This method uses the maximum principle applied to some quantities involving $|\nabla_x u^\varepsilon|^2$, and it requires that the functions u^ε are of class C^3. The latter is true here because of the smoothness assumptions on A and f.

¿From (2.36) and (2.38), and arguing as in [4] (Proposition 5.10), we obtain that u^ε converges (up to the extraction of some subsequence) almost everywhere in $\mathbb{R}\times\mathbb{R}^N$ to a function $u\in H^1_{loc}(\mathbb{R}\times\mathbb{R}^N)$ and

$$(u^\varepsilon, u^\varepsilon_t, \nabla_x u^\varepsilon) \rightharpoonup (u, u_t, \nabla_x u) \text{ in } L^2(\mathcal{K}),$$

for every compact subset $\mathcal{K}\subset\mathbb{R}\times\mathbb{R}^N$. From (2.36) and (2.38), one can actually assume that $u^\varepsilon\to u$ in $L^2_{loc}\left(\mathbb{R}\times\mathbb{R}^N\right)$ strong. Moreover, $0\leq u\leq p(x)$, $u_t\geq 0$ and from (2.36), (2.37) and (2.38),

$$\int_{\mathbb{R}\times\mathcal{K}_1} |\nabla_x u|^2+(u_t)^2 \leq C(\mathcal{K}_1), \tag{2.39}$$

for every compact subset $\mathcal{K}_1\subset\mathbb{R}^N$.

¿From parabolic regularity, u is then a classical solution of

$$\left\{\begin{array}{rcl} u_t-\nabla_x\cdot(A(x)\nabla_x u)-f(x,u) &=& 0 \text{ in } \mathbb{R}\times\mathbb{R}^N,\\ 0\leq u\leq p,\ u_t &\geq& 0 \text{ in } \mathbb{R}\times\mathbb{R}^N. \end{array}\right. \tag{2.40}$$

Moreover, one can assume, up to a translation in t, that

$$\forall\ \varepsilon>0,\ \int_{(0,1)\times C} u^\varepsilon(t,x+x_0)dxdt = |C|\frac{p^-}{2}. \tag{2.41}$$

for some $x_0\in\prod_{i=1}^N L_i\mathbb{Z}$ such that $x_0\cdot e>0$. Since $c^\varepsilon\to 0^+$ and u^ε is increasing in t, it follows that for all $B\in\mathbb{R}$, and for ε sufficiently small,

$$\forall\ (t,x)\in(0,1)\times C,\ u^\varepsilon(B+t,x)\leq u^\varepsilon(t+\frac{x_0\cdot e}{c^\varepsilon},x) = u^\varepsilon(t,x+x_0)$$

from (2.33). Next, one integrates over $(0,1)\times C$ and passes to the limit $\varepsilon\to 0^+$. By using (2.41) and the fact that $u^\varepsilon\rightharpoonup u$ in $L^2_{loc}\left(\mathbb{R}\times\mathbb{R}^N\right)$ weak,[4] one obtains

$$\forall\ B\in\mathbb{R},\ \int_{(0,1)\times C} u(B+t,x)dxdt \leq |C|\frac{p^-}{2}. \tag{2.42}$$

[4]This convergence is actually strong.

Using the monotonicity of u in t, let us define $u^+(x) = \lim_{t\to+\infty} u(t,x)$. Then, one has $u^+ \geq 0$ and $\nabla_x \cdot (A(x)\nabla_x u^+) + f(x,u^+) = 0$. Therefore, as it was stated in Theorems 2.1 and 2.3 of [8], $u^+ \equiv 0$ or $u^+ \equiv p$. But (2.41), passing to the limit $\varepsilon \to 0$ and $t \to +\infty$, rules out the case $u^+ \equiv 0$. Hence $u^+ \equiv p$. Next, using (2.42), and since u is nonincreasing in t, one obtains

$$\int_{(0,1)\times C} p(x)dx \leq |C|\frac{p^-}{2}$$

which is impossible. The proof of Proposition 2.9 is complete. □

2.4 Existence of a solution (c^1, u^1)

Let us choose a subsequence $\varepsilon \to 0$ such that $c^\varepsilon \to c^1 > 0$. For each ε, set $u^\varepsilon(t,x) = \phi^\varepsilon(x\cdot e + ct, x)$. As it was done in case 3 of Proposition 2.9, the functions u^ε converge (up to the extraction of some subsequence), in $H^1_{loc}(\mathbb{R}\times\mathbb{R}^N)$ weak, and almost everywhere, to a classical solution u^1 of (2.40).

Let us prove that $u^1\left(t + \frac{k\cdot e}{c^1}, x\right) = u^1(t, x+k)$ for all $k \in \prod_{i=1}^N L_i\mathbb{Z}$ and for all $(t,x) \in \mathbb{R}\times\mathbb{R}^N$. For all $B > 0$ and every compact set $\mathcal{K}_1$ in $\mathbb{R}^N$, one has,

$$\int_{(-B,B)\times\mathcal{K}_1} \left[u^\varepsilon\left(t + \frac{k\cdot e}{c^1}, x\right) - u^\varepsilon(t, x+k)\right]^2$$
$$= \int_{(-B,B)\times\mathcal{K}_1} \left[u^\varepsilon\left(t + \frac{k\cdot e}{c^1}, x\right) - u^\varepsilon\left(t + \frac{k\cdot e}{c^\varepsilon}, x\right)\right]^2,$$

whence

$$\int_{(-B,B)\times\mathcal{K}_1} \left[u^\varepsilon\left(t + \frac{k\cdot e}{c^1}, x\right) - u^\varepsilon(t, x+k)\right]^2$$
$$\leq \left(\frac{k\cdot e}{c^1} - \frac{k\cdot e}{c^\varepsilon}\right)^2 \int_{\mathbb{R}\times\mathcal{K}_1} (u^\varepsilon_t)^2 \leq \left(\frac{k\cdot e}{c^1} - \frac{k\cdot e}{c^\varepsilon}\right)^2 C(\mathcal{K}_1)$$

from (2.39). Therefore, by passing to the limit $\varepsilon \to 0$, we obtain

$$u^1\left(t + \frac{k\cdot e}{c^1}, x\right) = u^1(t, x+k) \tag{2.43}$$

almost everywhere in $\mathbb{R}\times\mathbb{R}^N$. Since u^1 is continuous, the equality holds for all $(t,x) \in \mathbb{R}\times\mathbb{R}^N$.

In order to obtain our result, one has to prove that $u^1(t,x) \to 0$ as $x\cdot e \to -\infty$ and $u^1(t,x) - p(x) \to 0$ as $x\cdot e \to +\infty$, locally in t. Since u^1 verifies (2.43), and $c > 0$, it is equivalent to prove that $u^1(t,x) \to 0$ as $t \to -\infty$ and $u^1(t,x) \to p(x)$ as $t \to +\infty$, locally in x.

As it was done in the proof of Proposition 2.9, case 3, one can assume, up to a translation in t, that

$$\forall\ \varepsilon,\ \int_{(0,1)\times C} u^\varepsilon(t,x)dxdt = |C|\frac{p^-}{2}. \tag{2.44}$$

Since u^1 is bounded and nondecreasing with respect to t, one can define $u^{\pm}(t,x) = \lim_{t\to\pm\infty} u^1(t,x)$. As done above, one knows that $u^{\pm}$ satisfies $\nabla\cdot(A(x)\nabla u^{\pm})+f(x,u^{\pm}) = 0$ and one has $0 \le u^{\pm} \le p$. As already said, this equation admits exactly two nonnegative solutions, which are not larger than p, namely 0 and p.

Passing to the limit $\varepsilon \to 0$ in (2.44), and using the fact that u^1 is nondecreasing with respect to t, one has

$$\int_C u^+(x) \ge |C|\frac{p^-}{2}, \tag{2.45}$$

and

$$\int_C u^-(x) \le |C|\frac{p^-}{2}. \tag{2.46}$$

One then easily concludes from (2.45) that u^+ is not equal to 0, and therefore $u^+ \equiv p$, and from (2.46), u^- is not equal to p, thus $u^- \equiv 0$.

¿From strong parabolic maximum principle, one obtains that u^1 is increasing in t. The existence result follows.

2.5 Existence of a solution (c,u) for all $c > c^1$

Proposition 2.11 *For each $c > c^1$, there exists a solution u of (1.10-1.11), associated to the speed c, and u is increasing in t.*

PROOF. Set $\phi^1(s,x) = u^1\left(\dfrac{s-x\cdot e}{c}, x\right)$, and, as before, define L_ε by

$$L_\varepsilon\phi = \nabla_x\cdot(A(x)\nabla_x\phi) + (eA(x)e+\varepsilon)\phi_{ss} + \nabla_x\cdot(A(x)e\phi_s) + \partial_s(eA(x)\nabla_x\phi) - c\phi_s.$$

Then, as it was done in [4] (Proposition 6.3), using Krylov-Safonov-Harnack type inequalities applied to $v = \partial_t u^1$, one gets the existence of a constant C such that $|\partial_{tt}u^1| \le C\partial_t u^1$ in $\mathbb{R}\times\mathbb{R}^N$, whence

$$L_\varepsilon\phi^1 + f(x,\phi^1) = \varepsilon\phi^1_{ss} + (c^1-c)\phi^1_s < 0 \text{ in } \mathbb{R}\times\mathbb{R}^N, \tag{2.47}$$

for $\varepsilon > 0$ small enough. In what follows, let $\varepsilon > 0$ be small enough so that (2.47) holds.

For any $a \in \mathbb{R}^+$ and $\tau \in \mathbb{R}$, set

$$h_\tau = \min_{\overline{C}} \phi^1(-a+\tau, .).$$

With a similar method as in Lemma 2.1, one can show the existence of a solution $\phi_\tau \in C^2\left(\overline{\Sigma_a}\right)$ of the following problem :

$$\begin{cases} L_\varepsilon\phi_\tau + f(x,\phi_\tau) &= 0 \text{ in } \Sigma_a, \\ \phi_\tau \text{ is L-periodic w.r.t. } x, & \\ \phi_\tau(-a,x) = h_\tau\dfrac{p(x)}{p^+},\ \phi_\tau(a,x) &= \phi^1(a+\tau,x) \text{ for all } x \in \mathbb{R}^N. \end{cases} \tag{2.48}$$

Let us now show that $h_\tau \dfrac{p(x)}{p^+} < \phi_\tau$ for all $(s,x) \in \Sigma_a$. First, since $f(x,u) = 0$ for all $u \leq 0$, one has, from the strong maximum principle, $\phi_\tau > 0$ in $\overline{\Sigma_a}$. Therefore, one can define

$$\gamma^* = \sup\left\{\gamma > 0,\ \phi_\tau > \gamma h_\tau \frac{p(x)}{p^+} \text{ in } \Sigma_a\right\}.$$

Assume that $\gamma^* < 1$. As in the proof of Lemma 2.2, using the fact that $\gamma^* h_\tau p(x)/p^+ < \phi_\tau(\pm a, x)$ for all $x \in \mathbb{R}^N$ (since $p(x)/p^+ \leq 1$ and ϕ^1 is increasing w.r.t. s), one gets the existence of $(s^*, x^*) \in (-a, a) \times C$ such that $\gamma^* h_\tau p(x)/p^+ \leq \phi_\tau(s, x)$ for all $(s, x) \in [-a, a] \times \mathbb{R}^N$, with equality at $(s^*, x^*) \in (-a, a) \times \mathbb{R}^N$. On the other hand,

$$L_\varepsilon(h_\tau \gamma^* \frac{p}{p^+}) = \gamma^* \frac{h_\tau}{p^+} L_\varepsilon(p) > -f(x, \gamma^* h_\tau \frac{p}{p^+}),$$

since $\gamma^* h_\tau / p^+ < 1$, and since $f(., s)/s$ is decreasing in s from our hypothesis on f. That leads to a contradiction as in Lemma 2.2.

Therefore, $\gamma^* \geq 1$, whence $\phi_\tau \geq h_\tau p/p^+$, and the strong maximum principle yields

$$\forall\ (s,x) \in \Sigma_a,\ h_\tau \frac{p(x)}{p^+} < \phi_\tau(s,x). \tag{2.49}$$

Similarly, one can easily show that $\phi_\tau(s, x) < p(x)$ for all $(s, x) \in \overline{\Sigma_a}$. Therefore, $\phi^1(s + \tau + k, x) \geq \phi_\tau(s, x)$ in $\overline{\Sigma_a}$ for k large enough. Let $\overline{k}$ be the smallest k such that the latter holds. From the boundary conditions in (2.48), one knows that $\overline{k} \geq 0$. Assume $\overline{k} > 0$. By continuity, it necessarily follows that $\phi^1(s + \tau + \overline{k}, x) \geq \phi_\tau(s, x)$ with equality at a point $(\overline{s}, \overline{x}) \in \overline{\Sigma_a}$. Since ϕ^1 is increasing in s,

$$\phi^1(-a + \tau + \overline{k}, \cdot) > \phi^1(-a + \tau, \cdot) \geq h_\tau \geq h_\tau \frac{p(\cdot)}{p^+} = \phi^\tau(-a, \cdot),$$

and $\phi^1(a + \tau + \overline{k}, \cdot) > \phi^1(a + \tau, \cdot) = \phi^\tau(a, \cdot)$. Therefore $(\overline{s}, \overline{x}) \in (-a, a) \times \overline{C}$ (one can assume this using the L-periodicity in x of ϕ^1 and ϕ_τ). But, from (2.47), it is found that $\phi^1(s + \tau + \overline{k}, x)$ is a supersolution of (2.48). Therefore, the strong maximum principle implies that $\phi^1(s+\tau+\overline{k}, x) = \phi_\tau(s, x)$ in $\overline{\Sigma_a}$. One gets a contradiction with the boundary condition at $s = a$. As a consequence, $\overline{k} = 0$ and one has

$$\forall\ (s,x) \in \overline{\Sigma_a},\ \phi_\tau(s,x) \leq \phi^1(s+\tau, x). \tag{2.50}$$

Since ϕ^1 is increasing in s, it also follows that $\phi_\tau(s, x) < \phi^1(a + \tau, x)$ in Σ_a.

As a conclusion, from (2.49) and (2.50), one has

$$\forall (s,x) \ \in \ \Sigma_a,\ h_\tau \frac{p(x)}{p^+} < \phi_\tau(s,x) < \phi^1(a+\tau, x).$$

Using the same sliding method as in Lemma 5.2 in [4], it follows that ϕ_τ is increasing in s and is the unique solution of (2.48) in $C^2\left(\overline{\Sigma_a}\right)$. Moreover, using the fact that the boundary conditions for ϕ_τ at $s = \pm a$ are increasing in τ, one can prove, as in Lemma 5.3

in [4], that the functions ϕ_τ are continuous with respect to τ in $C^2\left(\overline{\Sigma_a}\right)$ and increasing in τ. But, since $\phi^1(-\infty,x)=0$ and $\phi^1(+\infty,x)=p(x)$ in $\mathbb{R}^N$, it follows from (2.49) and (2.50) that $\phi_\tau \to 0$ as $\tau \to -\infty$ uniformly in $\overline{\Sigma_a}$ and that,

$$\forall\, \alpha>0,\ \exists\, T,\ \forall\, \tau>T,\quad \phi_\tau(s,x)>\frac{p^-}{p^+}p-\alpha \text{ in } \overline{\Sigma_a}.$$

Therefore, for each $a>1$, there exists a unique $\tau_\varepsilon(a)\in\mathbb{R}$ such that $\phi^{\varepsilon,a}:=\phi_{\tau_\varepsilon(a)}$ solves (2.48) and satisfies

$$\int_{(0,1)\times C}\phi^{\varepsilon,a}(s,x)dsdx=\frac{(p^-)^2}{2p^+}|C|\min(c,1). \tag{2.51}$$

Let $a_n\to+\infty$. From standard elliptic estimates, the functions ϕ^{ε,a_n} converge in $C^{2,\beta}_{loc}(\mathbb{R}\times\mathbb{R}^N)$ (for all $0\le\beta<1$), up to the extraction of a subsequence, to a function ϕ^ε satisfying

$$\left\{\begin{array}{rcl} L_\varepsilon\phi^\varepsilon+f(x,\phi^\varepsilon) &=& 0 \text{ in } \mathbb{R}\times\mathbb{R}^N,\\ \phi^\varepsilon \text{ is L-periodic w.r.t. } x. && \end{array}\right. \tag{2.52}$$

Moreover, ϕ^ε is nonincreasing with respect to s, satisfies $0\le\phi^\varepsilon(s,x)\le p(x)$ in $\mathbb{R}\times\mathbb{R}^N$, and

$$\int_{(0,1)\times C}\phi^\varepsilon(s,x)dsdx=\frac{(p^-)^2}{2p^+}|C|\min(c,1).$$

¿From standard elliptic estimates, and from the monotonicity of ϕ^ε with respect to s, one states that $\phi^\varepsilon(s,x)\to\phi^\varepsilon_\pm(x)$ as $s\to\pm\infty$ in $C^2(\overline{C})$. Moreover, $\phi^\varepsilon_\pm$ are L-periodic and satisfy

$$\nabla\cdot(A(x)\nabla\phi^\varepsilon_\pm)+f(x,\phi^\varepsilon_\pm)=0 \text{ in } \overline{C},$$

with $0\le\phi^\varepsilon_\pm(x)\le p(x)$ in $\overline{C}$.

But, as one has said before, from Theorems 2.1 and 2.3 of [8], the former equation, together with the bounds $0\le\phi^\varepsilon_\pm(x)\le p(x)$, admits exactly two nonnegative solutions, which are 0 and p. Since ϕ^ε is nondecreasing in s, and from (2.51), one has

$$\int_C\phi^\varepsilon_+(x)dx\ge\frac{(p^-)^2}{2p^+}|C|\min(c,1)>0, \tag{2.53}$$

and

$$\int_C\phi^\varepsilon_-(x)dx\le\frac{(p^-)^2}{2p^+}|C|\min(c,1)<\int_C p(x)dx. \tag{2.54}$$

¿From (2.53) one deduces that $\phi^\varepsilon_+\equiv p$ and from (2.54) one has $\phi^\varepsilon_-\equiv 0$.

Coming back to the original variables (t,x), one defines $u^\varepsilon(t,x)=\phi^\varepsilon(x\cdot e+ct,x)$. As it was done in the proof of (2.36) and Lemma 2.10, it follows from (2.52) and from the limiting behavior of ϕ^ε as $s\to\pm\infty$ that u^ε satisfies the estimates (2.39), independently of ε. As it was done in subsection 2.4, there exists a function $u\in H^1_{loc}(\mathbb{R}\times\mathbb{R}^N)$ such that (up to the extraction of a subsequence), $u^\varepsilon\rightharpoonup u$ weakly in $H^1_{loc}(\mathbb{R}\times\mathbb{R}^N)$. From parabolic regularity, u is then a classical solution of

$$\left\{\begin{array}{rcl} u_t-\nabla_x\cdot(A(x)\nabla_x u)-f(x,u) &=& 0 \text{ in } \mathbb{R}\times\mathbb{R}^N,\\ 0\le u\le p,\ u_t &\ge& 0 \text{ in } \mathbb{R}\times\mathbb{R}^N. \end{array}\right.$$

Moreover, as it was done in subsection 2.4, one still has $u(t+\frac{k\cdot e}{c},x)=u(t,x+k)$ in $\mathbb{R}\times\mathbb{R}^N$ for all $k\in\prod_{i=1}^N L_i\mathbb{Z}$. Furthermore, u satisfies

$$\int_{\{0<x\cdot e+ct<1,\ x\in C\}} u(t,x)dtdx=\frac{(p^-)^2}{2cp^+}|C|\min(c,1).$$

One deduces from standard parabolic estimates and from the monotonicity of u in t, that $u(t,x)\to u^\pm(x)$ locally in x as $t\to\pm\infty$, and that $u^\pm$ solve $\nabla\cdot(A(x)\nabla u^\pm)+f(x,u^\pm)=0$ in $\mathbb{R}^N$. Moreover, $0\le u^\pm\le p$. From the monotonicity of u with respect to t, one can also assert that

$$\int_C u^+(x)dx\ge\frac{(p^-)^2}{2cp^+}|C|\min(c,1)>0,$$

and

$$\int_C u^-(x)dx\le\frac{(p^-)^2}{2cp^+}|C|\min(c,1)<\int_C p(x)dx.$$

Therefore using the same argument as for $\phi_\pm^\varepsilon$, one concludes that $u^+\equiv p$ and $u^-\equiv 0$.

Finally, one deduces from the (t,x)-periodicity of u and the positivity of c that $u(t,x)\to 0$ as $x\cdot e\to-\infty$ and $u(t,x)-p(x)\to 0$ as $x\cdot e\to+\infty$, locally in t. Thus (c,u) is a classical solution of (1.10-1.11). Moreover, since $u_t\ge 0$, the strong parabolic maximum principle yields that u is increasing in t.

That completes the proof of Proposition 2.11. □

3 Monotonicity of the solutions

We are going to establish the monotonicity result in Theorem 1.2, namely, each solution (c,u) of (1.10-1.11) is such that u is increasing with respect to t. This will enable us to define a minimal speed c^* in the next section.

One first establishes the following lemma, which is close to Lemma 6.5 of [4]. Nevertheless, its proof does not use the fact that f has a given sign (which is not true in general) and clearly uses the property that 0 is an unstable solution of the stationary problem.

Lemma 3.1 *Let (c,u) be a classical solution of (1.10-1.11). Then $c>0$ and*

$$0<\Lambda\ :=\liminf_{t\to-\infty,\ x\in\overline{C}}\frac{u_t(t,x)}{u(t,x)}<+\infty.$$

PROOF. Let us first prove that $c>0$. Set $\phi(s,x)=u\left(\frac{s-x\cdot e}{c},x\right)$. From standard parabolic estimates, $u_t(t,x)\to 0$ as $t\to\pm\infty$, $\nabla_x u(t,x)\to 0$ as $t\to-\infty$ and $\nabla_x u(t,x)\to\nabla_x p(x)$ as $t\to+\infty$. Therefore $\phi_s(s,x)\to 0$ as $s\to\pm\infty$, $\nabla_x\phi(s,x)\to 0$ as $s\to-\infty$ and $\nabla_x\phi(s,x)\to\nabla_x p(x)$ as $s\to+\infty$. Then, arguing as in Lemma 2.5 (case 1), one can prove that

$$c\int_{\mathbb{R}\times C}(\phi_s)^2=-E(p)>0.$$

Therefore $c > 0$.

Next, as it was done in [4] (Lemma 6.5), one can assert, using standard interior estimates, Harnack type inequalities and the (t, x)-periodicity of u, that u_t/u and $\nabla u/u$ are globally bounded. Let Λ be defined as in the stating of the above lemma. Then Λ is a finite real number.

Let (t_n, x_n) be a sequence in $\mathbb{R} \times \overline{C}$ such that $t_n \to -\infty$ and

$$u_t(t_n, x_n)/u(t_n, x_n) \to \Lambda \text{ as } n \to +\infty.$$

Up to the extraction of some subsequence, one can assume that $x_n \to x_\infty \in \overline{C}$ as $n \to +\infty$. Now set

$$w_n(t, x) = \frac{u(t + t_n, x)}{u(t_n, x_n)}.$$

¿From the boundedness of u_t/u and $\nabla u/u$, one can assert that the functions w_n are locally bounded. Moreover, they satisfy

$$\partial_t w_n - \nabla \cdot (A(x)\nabla w_n) - \frac{f(x, u(t + t_n, x))}{u(t + t_n, x)} w_n = 0 \text{ in } \mathbb{R} \times \mathbb{R}^N.$$

¿From standard parabolic estimates, the positive functions w_n converge, up to the extraction of some subsequence, to a function w_∞, which is a nonnegative classical solution of

$$\partial_t w_\infty - \nabla \cdot (A(x)\nabla w_\infty) - f_u(x, 0)w_\infty = 0 \text{ in } \mathbb{R} \times \mathbb{R}^N,$$

since $u(t, x) \to 0$ as $t \to -\infty$ locally in x. Moreover, $w_\infty(0, x_\infty) = 1$, thus w_∞ is positive from the strong parabolic maximum principle. One also has $w_\infty(t + k \cdot e/c, x) = w_\infty(t, x + k)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ and for all $k \in \prod_{i=1}^N L_i\mathbb{Z}$.

Then using the arguments of the Lemma 6.5 of [4], one can check that the function $w_\infty(t, x)e^{-\Lambda t}$ does not depend on t. Indeed, one clearly has $(w_\infty)_t/w_\infty \geq \Lambda$, and $(w_\infty)_t(0, x_\infty) = \Lambda w_\infty(0, x_\infty)$ from the definition of (t_n, x_n). Therefore, the function $z = (w_\infty)_t/w_\infty$ satisfies

$$\partial_t z - \nabla \cdot (A(x)\nabla z) - 2\frac{\nabla w_\infty}{w_\infty} \cdot \nabla z = 0 \text{ in } \mathbb{R} \times \mathbb{R}^N$$

with $z \geq \Lambda$ and $z(0, x_\infty) = \Lambda$, and $\nabla w_\infty/w_\infty$ is bounded. The strong maximum principle yields $z \equiv \Lambda$; in other words, $w_\infty(t, x)e^{-\Lambda t}$ does not depend on t.

Therefore the $C^2(\mathbb{R}^N)$ function $\psi(x) = w_\infty(0, x)e^{-\Lambda(x\cdot e)/c}$ is positive and L-periodic. Moreover, it satisfies

$$-L_{c,\lambda}\psi = 0, \tag{3.55}$$

where one has set $\lambda = \Lambda/c$, and

$$\begin{aligned} -L_{c,\lambda}\psi \;=\; & -\nabla \cdot (A(x)\nabla\psi) - 2\lambda(eA(x)\nabla\psi) \\ & -[\lambda^2 eA(x)e + \lambda\nabla \cdot (A(x)e) - \lambda c + f_u(x, 0)]\psi. \end{aligned}$$

Now, from [4] (Proposition 5.7.1), one knows that for all λ and c in $\mathbb{R}$, there exists a unique $\mu_c(\lambda) \in \mathbb{R}$ and a unique positive function $\psi_\lambda \in C^2(\mathbb{R}^N)$ such that

$$\begin{cases} -L_{c,\lambda}\psi_\lambda = \mu_c(\lambda)\psi_\lambda \text{ in } \mathbb{R}^N, \\ \psi_\lambda \text{ is L-periodic}, \ \ \|\psi_\lambda\|_\infty = 1. \end{cases} \tag{3.56}$$

That allows us to define the function $\lambda \mapsto \mu_c(\lambda)$. Let us show that it is concave. First, using the result 2) of Proposition 5.7 in [4], one has

$$\mu_c(\lambda) = \max_{\phi \,\in\, E} \inf_{\mathbb{R}^N} \frac{-L_{c,\lambda}\phi}{\phi},$$

where $E = \left\{\phi \ \in \ C^2(\mathbb{R}^N), \ \phi > 0, \phi \ \text{ is } L-\text{periodic}\right\}$. Let E'_λ be the set defined by

$$E'_\lambda = \left\{\phi \ \in \ C^2(\mathbb{R}^N), \ \exists \ \Upsilon \ \in \ E \text{ with } \phi(x) = e^{\lambda x\cdot e}\Upsilon\right\}.$$

Then, $\mu_c(\lambda) = c\lambda + h(\lambda)$ with

$$h(\lambda) = \max_{\phi \,\in\, E'_\lambda} \inf_{\mathbb{R}^N} \left\{\frac{-\nabla\cdot(A(x)\nabla\phi)}{\phi} - f_u(x,0)\right\}.$$

Our aim is to show that h is concave. Let $\lambda_1, \ \lambda_2 \in \mathbb{R}$ and $t \in [0,1]$. Set $\lambda = t\lambda_1 + (1-t)\lambda_2$. One only has to show that $h(\lambda) \geq th(\lambda_1) + (1-t)h(\lambda_2)$. Let ϕ_1 and ϕ_2 be two arbitrary chosen functions in E'_{λ_1} and E'_{λ_2} respectively, and set $z_1 = \ln(\phi_1)$, $z_2 = \ln(\phi_2)$, $z = tz_1 + (1-t)z_2$ and $\phi = e^z$. It easily follows that $\phi \in E'_\lambda$. Therefore

$$h(\lambda) \geq \inf_{\mathbb{R}^N} \left\{\frac{-\nabla\cdot(A(x)\nabla\phi)}{\phi} - f_u(x,0)\right\}.$$

Moreover,

$$\frac{-\nabla\cdot(A(x)\nabla\phi)}{\phi} - f_u(x,0) = -\nabla\cdot(A(x)\nabla z) - \nabla z A(x)\nabla z - f_u(x,0),$$

and

$$\begin{aligned} \nabla z A(x)\nabla z \ &= \ t\nabla z_1 A(x)\nabla z_1 + (1-t)\nabla z_2 A(x)\nabla z_2 \\ &\quad -t(1-t)(\nabla z_1 - \nabla z_2)A(x)(\nabla z_1 - \nabla z_2) \\ &\leq \ t\nabla z_1 A(x)\nabla z_1 + (1-t)\nabla z_2 A(x)\nabla z_2, \end{aligned}$$

since $0 \leq t \leq 1$.

As a consequence,

$$\begin{aligned} \frac{-\nabla\cdot(A(x)\nabla\phi)}{\phi} - f_u(x,0) \ &\geq \ t[-\nabla\cdot(A(x)\nabla z_1) - \nabla z_1 A(x)\nabla z_1 - f_u(x,0)] \\ &\quad +(1-t)[-\nabla\cdot(A(x)\nabla z_2) - \nabla z_2 A(x)\nabla z_2 - f_u(x,0)] \\ &\geq \ t\left(\frac{-\nabla\cdot(A(x)\nabla\phi_1)}{\phi_1} - f_u(x,0)\right) \\ &\quad +(1-t)\left(\frac{-\nabla\cdot(A(x)\nabla\phi_2)}{\phi_2} - f_u(x,0)\right). \end{aligned}$$

Thus,

$$h(\lambda) \geq t \inf_{\mathbb{R}^N} \left(\frac{-\nabla \cdot (A(x)\nabla\phi_1)}{\phi_1} - f_u(x,0) \right) + (1-t) \inf_{\mathbb{R}^N} \left(\frac{-\nabla \cdot (A(x)\nabla\phi_2)}{\phi_2} - f_u(x,0) \right),$$

and, since ϕ_1 and ϕ_2 were arbitrarily chosen, one bets that $h(\lambda) \geq th(\lambda_1) + (1-t)h(\lambda_2)$. Therefore h is concave. This implies that h is continuous. Thus $\lambda \mapsto \mu_c(\lambda) = c\lambda + h(\lambda)$ is continuous and concave.

Next, let us show that $\mu_c(0) < 0$. By definition, $\mu_c(0)$ is the first eigenvalue of the linear problem $-\nabla \cdot (A(x)\nabla\psi)\psi - f_u(x,0)\psi$ with L-periodicity conditions. From the hypothesis for conservation, it follows that $\mu_c(0) < 0$.

Finally, it remains to show that $\mu_c'(0) > 0$. For each $\lambda \in \mathbb{R}$, consider the positive and L-periodic eigenfunction $\psi_\lambda \in C^2(\mathbb{R}^N)$ for problem (3.56), associated to the eigenvalue $\mu_c(\lambda)$. By definition, it satisfies the equation

$$\begin{array}{rcl} -\nabla \cdot (A(x)\nabla\psi_\lambda) - 2\lambda(eA(x)\nabla\psi_\lambda) & & \\ -(\lambda^2 eA(x)e + \lambda\nabla \cdot (A(x)e) - \lambda c + f_u(x,0))\psi_\lambda & = & \mu_1(\lambda)\psi_\lambda. \end{array}$$

Multiply this equation by ψ_0, and integrate it by parts over C. One obtains, using the L-periodicity of ψ_λ and ψ_0, and since the matrix field $A(x)$ is symmetric,

$$\begin{array}{rcl} -\int_C \psi_\lambda\nabla \cdot (A(x)\nabla\psi_0) - \lambda \int_C [\nabla \cdot (A(x)e\psi_\lambda) + eA(x)\nabla\psi_\lambda]\psi_0 & & \\ -\int_C \left[(\lambda^2 eA(x)e - \lambda c)\psi_\lambda + f_u(x,0)\psi_\lambda\right]\psi_0 & = & \mu_c(\lambda)\int_C \psi_\lambda\psi_0. \end{array} \tag{3.57}$$

Multiplying by ψ_λ the equation satisfied by ψ_0, one obtains,

$$-\int_C [\psi_\lambda\nabla \cdot (A(x)\nabla\psi_0) + f_u(x,0)\psi_\lambda\psi_0] = \mu_c(0)\int_C \psi_\lambda\psi_0. \tag{3.58}$$

Substituting (3.58) into (3.57), and dividing by λ, one gets

$$\begin{array}{rcl} -\int_C [\nabla \cdot (A(x)e\psi_\lambda) + eA(x)\nabla\psi_\lambda]\psi_0 & & \\ -\int_C (\lambda eA(x)e - c)\psi_\lambda\psi_0 & = & \dfrac{\mu_c(\lambda) - \mu_c(0)}{\lambda}\int_C \psi_\lambda\psi_0. \end{array} \tag{3.59}$$

Now, take an arbitrary sequence $\lambda_n \to 0$. Since $\mu_c(\lambda_n) \to \mu_c(0)$, standard elliptic estimates, and Sobolev injections imply, up to the extraction of some subsequence, that the functions ψ_{λ_n} converge locally (and therefore uniformly by L-periodicity) in $C^{2,\beta}$ (for all $0 \leq \beta < 1$) to a nonnegative function ψ^0 such that $\|\psi^0\|_\infty = 1$, ψ^0 is L-periodic and satisfies

$$-\nabla \cdot (A(x)\nabla\psi^0) - f_u(x,0)\psi^0 = \mu_c(0)\psi^0.$$

¿From strong elliptic maximum principle, it follows that $\psi^0 > 0$, and by uniqueness (up to normalization), $\psi^0 = \psi_0$, and the whole family ψ_{λ_n} converges to ψ_0 as $n \to +\infty$.

Therefore, passing to the limit $\lambda \to 0$ in (3.59), one obtains that μ_c is differentiable at 0, and

$$-\int_C [\nabla \cdot (A(x)e\psi_0) + eA(x)\nabla\psi_0]\psi_0 + c\int_C \psi_0^2 = \mu_c'(0)\int_C \psi_0^2.$$

¿From the L-periodicity of ψ_0, and since the matrix field $A(x)$ is symmetric, one has

$$\int_C [\nabla \cdot (A(x)e\psi_0) + eA(x)\nabla\psi_0]\psi_0 = \int_C [eA(x)\nabla\psi_0 - A(x)e \cdot \nabla\psi_0]\psi_0 = 0,$$

whence

$$\mu_c'(0) = c > 0.$$

Therefore, one has shown that $\lambda \mapsto \mu_c(\lambda)$ is concave, with $\mu_c(0) < 0$ and $\mu_c'(0) > 0$. Moreover, coming back to our solution ψ of (3.55), one has $\mu_c(\Lambda/c) = 0$. Therefore $\Lambda > 0$, and the lemma is proved. □

One can now turn to the proof of the monotonicity result in Theorem 1.2. Set $\phi(s,x) = u\left(\dfrac{s - x\cdot e}{c}, x\right)$. Then

$$u_t(t,x)/u(t,x) = c\phi_s(x\cdot e + ct, x)/\phi(x\cdot e + ct, x).$$

One knows from Lemma 3.1 that $c > 0$ and

$$\liminf_{t\to-\infty,\ x\in\overline{C}} \frac{u_t(t,x)}{u(t,x)} > 0.$$

Therefore,

$$\liminf_{s\to-\infty,\ x\in\mathbb{R}^N} \frac{\phi_s(s,x)}{\phi(s,x)} > 0$$

and, from the L-periodicity of ϕ with respect to x, one can deduce that there exists $\overline{s} \in \mathbb{R}$ such that

$$\forall\ s \le \overline{s},\ \forall\ x \in \mathbb{R}^N,\ \phi_s(s,x) > 0.$$

Moreover $\inf_{s\ge\overline{s},\ x\in\mathbb{R}^N} \phi(s,x) > 0$ and $\phi(-\infty,x) = 0$ uniformly in $x \in \mathbb{R}^N$. As a consequence, there exists $B \in \mathbb{R}$ such that $-B \le \overline{s}$ and

$$\forall\ \tau \ge 0,\ \forall\ s \le -B,\ \forall\ x \in \mathbb{R}^N,\ \phi(s,x) \le \phi^\tau(s,x) \tag{3.60}$$

where one has defined $\phi^\tau(s,x) = \phi(s+\tau,x)$. One can assume that $B \ge 0$.

Fix now any $\tau \ge 0$. Set

$$\lambda^* = \inf\left\{\lambda,\ \lambda\phi^\tau \ge \phi \text{ in } [-B,+\infty)\times\mathbb{R}^N\right\}.$$

The real λ^* is well defined since ϕ is bounded and $\inf_{s\ge -B,\ x\in\mathbb{R}^N} \phi^\tau(s,x) > 0$.

Assume $\lambda^* > 1$. Since $\phi(s,x) \to p(x) > 0$ as $s \to +\infty$ uniformly in x, with p bounded from below, and since ϕ is L-periodic in x, there exists a point $(s_0,x_0) \in [-B,+\infty)\times\overline{C}$ such that $\lambda^*\phi^\tau(s_0,x_0) = \phi(s_0,x_0)$.

Furthermore, $\lambda^*\phi^\tau \geq \phi$ in $\in [-B, +\infty) \times \mathbb{R}^N$ by continuity and in $(-\infty, -B] \times \mathbb{R}^N$ by (3.60) and because $\lambda^* > 1$. Coming back to the original variables (t, x), set $z(t, x) = \lambda^*\phi^\tau(x \cdot e + ct, x) - \phi(x \cdot e + ct, x)$. Then $z \geq 0$ in $\mathbb{R} \times \mathbb{R}^N$, moreover, z satisfies the following equation :

$$z_t - \nabla \cdot (A(x)\nabla z) = \lambda^* f(x, \phi^\tau) - f(x, \phi).$$

Therefore, using (1.7), one obtains

$$f(x, \lambda^*\phi^\tau) \leq \lambda^* f(x, \phi^\tau).$$

Thus one has

$$z_t - \nabla \cdot (A\nabla z) \geq f(x, \lambda^*\phi^\tau) - f(x, \phi).$$

Therefore, there exists a bounded function b such that

$$z_t - \nabla \cdot (A(x)\nabla z) + b(x)z \geq 0. \tag{3.61}$$

Furthermore, since $\lambda^*\phi_\tau(s_0, x_0) = \phi(s_0, x_0)$, setting $t_0 = \frac{s_0 - x_0 \cdot e}{c}$, one has $z(t_0, x_0) = 0$. Besides, z is nonnegative, and satisfies (3.61); therefore, from the strong parabolic maximum principle, one has $z(t, x) = 0$ for all $t \leq t_0$ and $x \in \mathbb{R}^N$, whence $z(t, x) \equiv 0$ in $\mathbb{R} \times \mathbb{R}^N$ since $z\left(t + \frac{k \cdot e}{c}, x\right) = z(t, x + k)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ and $k \in \prod_{i=1}^N L_i\mathbb{Z}$. One gets a contradiction since $z(t, x) \to (\lambda^* - 1)p(x) > 0$ as $t \to +\infty$.

Thus $\lambda^* \leq 1$ for all $\tau \geq 0$, whence $\phi^\tau \geq \phi$ in $\mathbb{R} \times \mathbb{R}^N$ for all $\tau \geq 0$. One therefore gets that ϕ is nondecreasing with respect to s, and u is nondecreasing in t because $c > 0$. Finally, with the same arguments as above, one can prove, using the strong parabolic maximum principle, that u is increasing in t. That concludes the proof of the monotonicity result in Theorem 1.2. □

Remark 3.2 *Notice that this monotonicity result especially implies that any solution* (c, u) *of (1.10-1.11) is such that* $0 < u(t, x) < p(x)$ *for all* $(t, x) \in \mathbb{R} \times \mathbb{R}^N$.

4 The minimal speed c^*

This section is devoted to the proof of the existence of a minimal speed of propagation of the pulsating fronts, and some properties of this minimal speed with respect to the nonlinearity f.

4.1 Existence of a positive minimal speed c^*

As it was proved in Lemma 3.1, any solution (c, u) of (1.10-1.11) is such that $c > 0$. In order to complete the proof of the first part of Theorem 1.2 and to obtain the existence of a $c^* > 0$ such that there exists a solution of (1.10-1.11) if and only if $c \geq c^*$.

Proposition 4.1 *There exists* $c^* > 0$ *such that, for* $c \geq c^*$*, there exists a solution* u *of (1.10-1.11), while no solution exists for* $c < c^*$.

PROOF. First, assume by contradiction that there exists a sequence $c_n \to 0^+$ and some classical functions u_n such that (c_n, u_n) is a solution of (1.10-1.11).

As already done, let $x_0 \in \prod_{i=1}^N L_i\mathbb{Z}$, be such that $x_0 \cdot e > 0$. One can assume that $u_n(0, x_0) = \dfrac{p^-}{2}$.

¿From standard parabolic estimates, the positive functions u_n converge locally uniformly, up to the extraction of some subsequence, to a nondecreasing (in t) function u, which is a classical solution of

$$\partial_t u - \nabla \cdot (A(x)\nabla u) = f(x, u) \ \in \mathbb{R} \times \mathbb{R}^N.$$

Moreover, u satisfies $0 \le u \le p$ and one has $u(0, x_0) = \dfrac{p^-}{2}$.

Since u is nondecreasing in t, one can define $u^+(x) =: \lim\limits_{t\to+\infty} u(t, x)$, and from standard elliptic estimates, u^+ satisfies $\nabla \cdot (A(x)\nabla u^+) + f(x, u^+) = 0$. Moreover $0 \le u^+ \le p$. Hence, as already said (using Theorems 2.1 and 2.3 of [8]), $u^+ \equiv 0$ or $u^+ \equiv p$. But for every $B > 0$, for n large enough, $u_n(B, 0) \le u_n(\frac{x_0 \cdot e}{c_n}, 0) = u_n(0, x_0) = \frac{p^-}{2}$. Therefore $u^+(0) \le \frac{p^-}{2}$. Thus $u^+ \equiv 0$. But since u is nondecreasing and $u(0, x_0) = \frac{p^-}{2}$, $u^+(0) \ge \frac{p^-}{2}$, which is contradictory with the preceding result.

On the other hand, the arguments used in Proposition 2.11 actually imply that, if (c_0, u_0) is a solution of (1.10-1.11) with $c_0 > 0$ and $(u_0)_t > 0$, then there is a solution (c, u) of (1.10-1.11) for each $c > c_0$.

Using Lemma 3.1, one concludes that there exits $c^* > 0$ such that for all $c > c^*$, there exists a solution u of (1.10-1.11), while no solution exists for $c < c^*$.

In particular, there exists a sequence (c_n, u_n) of solutions of (1.10-1.11), such that $c_n \to c^*$ as $n \to +\infty$, with $c_n > c^*$. As it was done in the first part of the proof of this Proposition assume that $u_n(0, x_0) = \frac{p^-}{2}$. From standard parabolic estimates, u_n converge locally uniformly in (up to the extraction of some subsequence), to a classical solution u^* of

$$\partial_t u^* - \nabla \cdot (A(x)\nabla u^*) = f(x, u^*) \ \in \mathbb{R} \times \mathbb{R}^N,$$

with $0 \le u^* \le p$, and $u^*_t \ge 0$. Moreover, $u^*(0, x_0) = \frac{p^-}{2}$. Using the same arguments as those of the beginning of this proof, one concludes that $\lim_{t\to-\infty} u^*(t, x) = 0$ and $\lim_{t\to+\infty} u^*(t, x) = p(x)$, locally in x. Furthermore, by passing to the limit, $u^*\left(t + \frac{k \cdot e}{c^*}, x\right) = u^*(t, x + k)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ and $k \in \prod_{i=1}^N L_i\mathbb{Z}$. Finally, the strong maximum principle, with $u^*_t \ge 0$, gives us that u^* is increasing in t. □

4.2 Characterization of c^*

This section is devoted to the proof of the variational characterization of the minimal speed c^*. Notice first that the assumption (1.7) implies that

$$\forall\, x \in \mathbb{R}^N,\ \forall\, u \ge 0,\ f(x, u) \le f_u(x, 0)u. \tag{4.62}$$

Let us define

$$c_0^* = \inf \{c \in \mathbb{R},\ \exists\, \lambda > 0 \text{ with } \mu_c(\lambda) = 0\},$$

where $\mu_c(\lambda)$ is the principal eigenvalue of the elliptic operator

$$\begin{array}{rl} -L_{c,\lambda}\psi = & -\nabla\cdot(A(x)\nabla\psi) - 2\lambda\ eA(x)\nabla\psi \\ & -[\lambda^2 eA(x)e + \lambda\nabla\cdot(A(x)e) - \lambda c + f_u(x,0)]\psi, \end{array}$$

with L-periodicity conditions.

Proposition 4.2 *One has $c^* = c_0^*$.*

The proof is divided into several lemmas.

Lemma 4.3 *The real number c_0^* does exist and $0 \le c_0^* \le c^*$.*

PROOF. Let $c \geq c^*$, and (c,u) be a solution of (1.10-1.11). Then, arguing as in the proof of Lemma 3.1, one obtains a positive function ψ, satisfying (3.55) with $\lambda = \Lambda/c > 0$. In other words, $\mu_c(\lambda) = 0$. That yields $c_0^* \le c^*$.

Moreover, using the concavity of $\lambda \mapsto \mu_c(\lambda)$, which has been shown in the proof of Lemma 3.1, together with $\mu_c(0) < 0$ and $(\mu_c)'(0) = c$, one immediately gets that if $c < 0$, then $\mu_c(\lambda) < 0$ for all $\lambda > 0$. Therefore 0 is a lower bound of the set $\{c \in \mathbb{R},\ \exists\ \lambda > 0$ with $\mu_c(\lambda) = 0\}$. □

¿From Lemma 4.3 and Proposition 4.1, the next lemma follows :

Lemma 4.4 *For all $c < c_0^*$, problem (1.10-1.11) has no solution (c,u).*

Now, for all $\varepsilon > 0$, let us define

$$c_\varepsilon^* = \inf\left\{c \in \mathbb{R},\ \exists\ \lambda > 0 \text{ with } \mu_c^\varepsilon(\lambda) = 0\right\},$$

where $\mu_c^\varepsilon(\lambda)$ is the principal eigenvalue of the elliptic operator

$$\begin{array}{rl} -L_{c,\lambda}^\varepsilon\psi = & -\nabla\cdot(A(x)\nabla\psi) - 2\lambda eA(x)\nabla\psi \\ & -(eA(x)e+\varepsilon)\lambda^2\psi - \lambda\nabla\cdot(A(x)e)\psi + \lambda c\psi - f_u(x,0)\psi, \end{array}$$

with L-periodicity conditions.

First, using a result of [4] (Proposition 5.7.2), one obtains that

$$\mu_c^\varepsilon(\lambda) = \max_{\phi\in E}\ \inf_{\mathbb{R}^N}\ \frac{-L_{c,\lambda}^\varepsilon\phi}{\phi},$$

where $E = \left\{\phi \in C^2(\mathbb{R}^N),\ \phi > 0,\ \phi \text{ is L-periodic}\right\}$.

Set

$$j(\lambda) = \max_{\phi\in E}\ \inf_{x\in\mathbb{R}^N}\ \left\{\frac{-\nabla\cdot(A(x)\nabla\phi) - 2\lambda eA(x)\nabla\phi}{\phi} - \lambda^2 eA(x)e - \lambda\nabla\cdot(A(x)e) - f_u(x,0)\right\}.$$

Then,

$$\mu_c^\varepsilon(\lambda) = j(\lambda) + \lambda c - \varepsilon\lambda^2 = \mu_c(\lambda) - \varepsilon\lambda^2.$$

Lemma 4.5 *The real number c_ε^* does exist for all $\varepsilon > 0$, and $c_\varepsilon^* \geq 0$.*

PROOF. Let ε be fixed. Let λ be given. Since $\mu_c^\varepsilon(\lambda) = j(\lambda) + \lambda c - \varepsilon\lambda^2$, there exists $c > 0$ large enough such that $\mu_c^\varepsilon(\lambda) > 0$. Since $\mu_c^\varepsilon(0) = \lambda_1 < 0$ (λ_1 is the first eigenvalue of $-\nabla \cdot (A(x)\nabla\psi) - f_u(x,0)\psi$ with L-periodicity conditions), and since $\lambda \mapsto \mu_c^\varepsilon(\lambda) = \mu_c(\lambda) - \varepsilon\lambda^2$ is concave, whence continuous, one gets the existence of λ' such that $\mu_c^\varepsilon(\lambda') = 0$. Therefore $c_\varepsilon^* < +\infty$ for all $\varepsilon > 0$.

Moreover, as it was done in Lemma 4.3, one easily sees that 0 is a lower bound of the set $\{c \in \mathbb{R},\ \exists\ \lambda > 0 \text{ with } \mu_c^\varepsilon(\lambda) = 0\}$. □

Let us now show that

Lemma 4.6 *For all $c > c_\varepsilon^*$, there exists $\lambda > 0$ such that $\mu_c^\varepsilon(\lambda) = 0$.*

PROOF. Let c be s.t. $c > c_\varepsilon^*$. From the definition of c_ε^*, one knows that there exists a sequence (c_n) such that $c_n \to c_\varepsilon^*$ as $n \to +\infty$ and, for each n, there is $\lambda_n > 0$ with $\mu_c^\varepsilon(\lambda_n) = 0$. Therefore, there exists N such that $c_N < c$. One has $\mu_c^\varepsilon(\lambda_N) = \mu_{c_N}^\varepsilon(\lambda_N) + (c - c_N)\lambda_N > 0$. Using the same argument than this of Lemma 4.5, one deduces that there exists $\lambda > 0$ such that $\mu_c^\varepsilon(\lambda) = 0$. □

Next, let us prove that

Lemma 4.7 *One has $c_\varepsilon^* \to c_0^*$ as $\varepsilon \to 0$.*

PROOF. First, one observes that $c_\varepsilon^* \geq c_0^*$ for all $\varepsilon > 0$. Indeed, for $c > c_\varepsilon^*$, there exists, from Lemma 4.6, $\lambda > 0$ such that $\mu_c^\varepsilon(\lambda) = 0$. Thus, since $\mu_c(\lambda) > \mu_c^\varepsilon(\lambda) = 0$, arguing as in Lemma 4.6, one easily sees that there exists $\lambda_0 > 0$ such that $\mu_c(\lambda_0) = 0$.

Next, let us show that for any $c > c_0^*$, there exists ε_0 such that $c \geq c_\varepsilon^*$ for all $\varepsilon < \varepsilon_0$.

Indeed, one deduces from Lemma 4.6, adapted to c_0^*, that for each $c > c_0^*$, there exists $\lambda_1 > 0$ such that $\mu_{\frac{c+c_0^*}{2}}(\lambda_1) = 0$. Then

$$\mu_c^\varepsilon(\lambda_1) = \mu_{\frac{c+c_0^*}{2}}(\lambda_1) + \frac{c - c_0^*}{2}\lambda_1 - \lambda_1^2\varepsilon.$$

Thus, $\mu_c^\varepsilon(\lambda_1) = \frac{c-c_0^*}{2}\lambda_1 - \lambda_1^2\varepsilon$. Hence, for ε small enough, $\mu_c^\varepsilon(\lambda_1) > 0$. Therefore, there exists $\lambda > 0$ such that $\mu_c^\varepsilon(\lambda) = 0$. Finally, from the definition of c_ε^*, one deduces that $c \geq c_\varepsilon^*$, and the lemma is proved. □

Let us now turn to the

PROOF of Proposition 4.2. Let c be such that $c > c_0^*$. Then, from Lemma 4.7, one knows that for ε small enough, $c > c_\varepsilon^*$. Therefore, from Lemma 4.6, there exist $\lambda > 0$ and $\psi > 0$ L-periodic, depending on ε, and such that

$$-L_{c,\lambda}^\varepsilon\psi = 0. \tag{4.63}$$

Now, set $\phi^1(s,x) := \psi(x)e^{\lambda s}$, for all $(s,x) \in \mathbb{R} \times \mathbb{R}^{\mathbb{N}}$, and let L_ε be defined as in the proof of Proposition 2.11. Then,

$$\begin{aligned} L_\varepsilon \phi^1 = \ & \{ \nabla \cdot (A(x)\nabla\psi) + 2\lambda e A(x)\nabla\psi \\ & +(eA(x)e+\varepsilon)\lambda^2\psi + \lambda\nabla\cdot(A(x)e)\ \psi - \lambda c\psi\}\, e^{\lambda s}, \end{aligned}$$

and, since ψ satisfies (4.63), one has

$$L_\varepsilon \phi^1 = -f_u(x,0)\phi^1.$$

Therefore, using (4.62), one obtains,

$$L_\varepsilon\phi^1 + f(x,\phi^1) = f(x,\phi^1) - f_u(x,0)\phi^1 \leq 0. \tag{4.64}$$

Moreover, ϕ^1 is increasing in s and L-periodic with respect to x.

Now, with the notations of Section 2.1, and as it was proved in Lemma 2.1, there exists a solution $\phi_\tau \in C^2\left(\overline{\Sigma_a}\right)$ of the following problem :

$$\begin{cases} L_\varepsilon\phi_\tau + f(x,\phi_\tau) = 0 \text{ in } \Sigma_a, \\ \phi_\tau \text{ is L-periodic in } x, \\ \phi_\tau(-a,x) = \min\left\{ \inf\limits_{y\in\overline{C}} \phi^1(-a+\tau,y), p^- \right\} p(x)/p^+, \\ \phi_\tau(a,x) = \min\{\phi^1(a+\tau,x), p(x)\}. \end{cases} \tag{4.65}$$

First, following the proof of Proposition 2.11, one obtains that

$$\forall\ (s,x)\ \in\ \Sigma_a,\ \ \phi_\tau(-a,x) < \phi_\tau(s,x). \tag{4.66}$$

Next, using $\phi_\tau(a,x) = \min\{\phi^1(a+\tau,x),p(x)\}$, and since $\phi^1(s,x) \to +\infty$ as $s \to +\infty$, one has, for k large enough, $\phi^1(s+\tau+k,x) > \phi_\tau(s,x)$ in $\overline{\Sigma_a}$. Let $\overline{k}$ be the smallest k such that the latter holds. It exists since $\phi^1(s,x) \to 0$ as $s \to -\infty$ and $\phi_\tau(s,x) > 0$ in $\overline{\Sigma_a}$. Assume $\overline{k} > 0$. By continuity, $\phi^1(s+\tau+\overline{k},x) \geq \phi_\tau(s,x)$ with equality at a point $(\overline{s},\overline{x}) \in \overline{\Sigma_a}$. Since ϕ^1 is increasing in s, $\phi^1(-a+\tau+\overline{k},x) > \phi^1(-a+\tau,x) \geq \phi_\tau(-a,x)$ in $\overline{C}$, and similarly, $\phi^1(a+\tau+\overline{k},x) > \phi^1(a+\tau,x) \geq \phi_\tau(a,x)$. Thus, $(\overline{s},\overline{x}) \in (-a,a)\times\overline{C}$ (one can assume this using the L-periodicity in x of ϕ^1 and ϕ_τ). But, from (4.64), it is found that $\phi^1(s+\tau+\overline{k},x)$ is a super-solution of (4.65). Therefore, the strong maximum principle implies that $\phi^1(s+\tau+\overline{k},x) \equiv \phi_\tau(s,x)$ in $\overline{\Sigma_a}$. One gets a contradiction with the boundary condition at $s=a$. As a consequence, $\overline{k}=0$ and, one has

$$\forall\ (s,x) \in \overline{\Sigma_a},\ \ \phi_\tau(s,x) \leq \phi^1(s+\tau,x) \tag{4.67}$$

and, since ϕ^1 is increasing in s, it follows that $\phi_\tau(s,x) < \phi^1(a+\tau,x)$ in Σ_a.

As a conclusion, from (4.66) and (4.67), one has

$$\forall\ (s,x) \in \Sigma_a,\ \ \phi_\tau(-a,x) < \phi_\tau(s,x) < \phi_\tau(a,x). \tag{4.68}$$

Using the same arguments as those of Proposition 2.11, it follows that ϕ_τ is increasing in s and is the unique solution of (4.65) in $C^2\left(\overline{\Sigma_a}\right)$. Moreover, since the boundary conditions for ϕ_τ at $s = \pm a$ are nondecreasing in τ, one can prove, as in Lemma 2.3, that the functions ϕ_τ are continuous with respect to τ in $C^2\left(\overline{\Sigma_a}\right)$ and nondecreasing in τ. But, since $\phi^1(-\infty, x) = 0$ and $\phi^1(+\infty, x) = +\infty$ in $\mathbb{R}^N$, it follows from (4.68) that $\phi_\tau \to 0$ as $\tau \to -\infty$ uniformly in $\overline{\Sigma_a}$ and that,

$$\forall\ \alpha > 0,\ \exists\ T,\ \forall \tau > T, \quad \phi_\tau > \frac{p^- p}{p^+} - \alpha \text{ in } \overline{\Sigma_a}.$$

Therefore, for each $a > 1$, there exists $\tau(a) \in \mathbb{R}$ such that $\phi^{\varepsilon,a} := \phi_{\tau(a)}$ solves (4.65) and satisfies

$$\int_{(0,1)\times C} \phi^{\varepsilon,a}(s,x)dsdx = \frac{(p^-)^2}{2p^+}|C| \min(c,1).$$

Moreover $\phi_\tau(a,.)$ is bounded independently of a. Thus, letting $a_n \to +\infty$, from standard elliptic estimates, the functions ϕ^{ε,a_n} converge in $C^{2,\beta}_{loc}(\mathbb{R}\times\mathbb{R}^N)$ (for all $0 \le \beta < 1$), up to the extraction of a subsequence, to a function ϕ^ε satisfying

$$\begin{cases} L_\varepsilon \phi^\varepsilon + f(x,\phi^\varepsilon) \ = \ 0 \text{ in } \mathbb{R}\times\mathbb{R}^N, \\ \phi^\varepsilon \text{ is L-periodic w.r.t. } x. \end{cases}$$

Moreover, ϕ^ε is nonincreasing with respect to s, and satisfies

$$0 \le \phi^\varepsilon(s,x) \le p(x) \text{ in } \mathbb{R}\times\mathbb{R}^N,$$

and

$$\int_{(0,1)\times C} \phi^{\varepsilon}(s,x)dsdx = \frac{(p^-)^2}{2p^+}|C| \min(c,1).$$

Next, passing to the limit $\varepsilon \to 0$ and using the same arguments as those of the end of the proof of Proposition 2.11, one obtains a solution (c,u) of the problem (1.10-1.11).

But since c was chosen arbitrarily such that $c > c_0^*$, one concludes that there exists a solution (c,u) of (1.10-1.11) for all $c > c_0^*$. Next, using Lemma 4.4, one obtains that $c^* = c_0^*$. $\Box$

That completes the proof of Theorem 1.2.

4.3 Dependency of c^* with respect to the nonlinearity f

In this section, we study the dependency of c^*, with respect to the "shape" and the "size" of the nonlinearity f. This section is devoted to the proofs of Theorem 1.3 and Corollary 1.4. In the whole section, one assumes that the matrix field $A(x) = A$ is constant, and one considers the problem (1.10-1.11), with a nonlinearity f such that $f_u(x,0)$ is of the type $f_u(x,0) = \mu(x) + B\nu(x)$, where B is a positive real number and ${}_m u$ and ν are periodic $C^{0,\alpha}$ functions.

It then follows from Theorem 2.8 of [8] that the function f satisfies the hypothesis for all $B > 0$ if $\displaystyle\int_C \mu \ge 0$ and $\displaystyle\int_C \nu \ge 0$ with $\nu \not\equiv 0$. It also follows from Theorem 2.8 of [8] that f satisfies the hypothesis for conservation for $B > 0$ large enough under the only assumption $\max \nu > 0$.

In the next propositions, we will make several uses of the following characterizations of c^* : first, from Theorem 1.2,

$$c^* = \inf \{c,\ \exists\ \lambda > 0 \text{ with } \mu_{c,B}(\lambda) = 0\}, \tag{4.69}$$

where $\mu_{c,B}(\lambda)$ is the principal eigenvalue of the elliptic operator

$$-L_{c,B,\lambda}\psi = -\nabla \cdot (A\nabla\psi) - 2\lambda Ae \cdot \nabla\psi - (\lambda^2\ eAe - \lambda c)\psi - (\mu(x) + B\nu(x))\psi,$$

on the set E of L-periodic C^2 functions. Furthermore, as it was said in [4] and in [30], for pulsating fronts in $\mathbb{R}^N$, the formula below is equivalent to the following one :

$$c^* = \min_{\lambda>0} \frac{-k_\lambda(B)}{\lambda}, \tag{4.70}$$

where $k_\lambda(B)$ is the principal eigenvalue of the operator

$$-\mathcal{L}_{B,\lambda}\phi = -\nabla \cdot (A\nabla\phi) - 2\lambda Ae \cdot \nabla\phi - \lambda^2\ eAe\phi - (\mu(x) + B\nu(x))\phi,$$

acting on the same set E of functions ϕ. We call $\phi_{B,\lambda}$ be the principal eigenfunction associated to $k_\lambda(B)$. It satisfies

$$\begin{cases} -\mathcal{L}_{B,\lambda}\phi_{B,\lambda} = k_\lambda(B)\phi_{B,\lambda}, \\ \phi_{B,\lambda} \text{ is L-periodic, } \phi_{B,\lambda} > 0 \text{ in } \mathbb{R}^N, \\ \|\phi_{B,\lambda}\|_\infty = 1 \text{ (up to normalization).} \end{cases} \tag{4.71}$$

We are going to study the monotonicity of the function $B \mapsto c^* = c^*(B)$, as soon as the hypothesis for conservation if satisfied. One has the

Proposition 4.8 *Assume that* $\mu = \mu_0 \geq 0$ *is constant and assume that* $\displaystyle\int_C \nu(x)dx \geq 0$ *with* $\max \nu > 0$. *Then, the hypothesis for conservation is satisfied for all* $B > 0$ *and* $c^*(B)$ *is an increasing function of* $B > 0$.

PROOF. As already underlined at the beginning of this section, the hypothesis for conservation is satisfied for all $B > 0$.

As done in the proof of Lemma 3.1, one has

$$k_\lambda(B) = \max_{\phi \in E'_\lambda} \inf_{\mathbb{R}^N} \frac{-\nabla \cdot (A\nabla\phi)}{\phi} - \mu_0 - B\nu(x),$$

where E'_λ be the set defined by

$$E'_\lambda = \left\{\phi \in C^2(\mathbb{R}^N),\ \exists\ \Upsilon > 0,\ \Upsilon\ L\text{-periodic with } \phi(x) = e^{\lambda x \cdot e}\Upsilon\right\}.$$

Let $B_1,\ B_2 \in \mathbb{R}$ and $t \in [0,1]$. Set $B = tB_1 + (1-t)B_2$. Let ϕ_1 and ϕ_2 be two arbitrary chosen functions in E'_λ, and set $z_1 = \ln(\phi_1)$, $z_2 = \ln(\phi_2)$, $z = tz_1 + (1-t)z_2$ and $\phi = e^z$. It easily follows that $\phi \in E'_\lambda$. Therefore

$$k_\lambda(B) \geq \inf_{\mathbb{R}^N} \left\{ \frac{-\nabla \cdot (A\nabla\phi)}{\phi} - \mu_0 - B\nu(x) \right\}.$$

Then, arguing as in the proof of Lemma 3.1, one obtains that

$$\begin{aligned} k_\lambda(B) \ \geq \ & t\inf_{\mathbb{R}^N} \left\{ \frac{-\nabla \cdot (A\nabla\phi_1)}{\phi_1} - \mu_0 - B_1\nu(x) \right\} \\ & +(1-t)\inf_{\mathbb{R}^N} \left\{ \frac{-\nabla \cdot (A\nabla\phi_2)}{\phi_2} - \mu_0 - B_2\nu(x) \right\}, \end{aligned}$$

and, since ϕ_1 and ϕ_2 were arbitrary chosen, one has $k_\lambda(B) \geq tk_\lambda(B_1) + (1-t)k_\lambda(B_2)$. Therefore the function $B \mapsto k_\lambda(B)$ is concave. This also implies that this function is continuous.

Next, one easily sees that $k_\lambda(0) = -\lambda^2 eAe - \mu_0$, and that the associated eigenfunction $\phi_{0,\lambda}$ is equal to 1.

Now, let us calculate $k'_\lambda(0)$. Let $\phi_{B,\lambda}$ be the principal eigenfunction associated to $k_\lambda(B)$ defined in (4.71), and let us integrate by parts the equation (4.71) over C. Using the L-periodicity of $\phi_{B,\lambda}$, one obtains

$$-(\lambda^2 eAe + \mu_0)\int_C \phi_{B,\lambda} - B\int_C \nu(x)\phi_{B,\lambda}dx = k_\lambda(B)\int_C \phi_{B,\lambda}. \tag{4.72}$$

By continuity, one knows that $k_\lambda(B) \to k_\lambda(0)$ as $B \to 0$. Still arguing as in the proof of Lemma 3.1, one also knows that $\phi_{B,\lambda}$ converges in $C^{2,\beta}$ (for all $0 \leq \beta < 1$) to $\phi_{0,\lambda} \equiv 1$ as $B \to 0$. Then, dividing the equation (4.72) by B, one gets

$$\frac{k_\lambda(B) + \lambda^2 eAe + \mu_0}{B}\int_C \phi_{B,\lambda} = -\int_C \nu(x)\phi_{B,\lambda}dx.$$

Therefore, passing to the limit $B \to 0$, one obtains

$$k'_\lambda(0) = -\int_C \nu(x)dx.$$

In the case $\displaystyle\int_C \nu(x)dx > 0$, one has $k'_\lambda(0) < 0$. From the concavity of $B \mapsto k_\lambda(B)$, one deduces that this function is decreasing with respect to $B > 0$. Since this is true for all $\lambda > 0$, one concludes that the minimal speed $c^*(B)$ given in (4.70) is an increasing function of $B > 0$.

Similarly, if $\displaystyle\int_C \nu(x)dx = 0$, and $\max\nu > 0$, divide the equation (4.71) by $\phi_{B,\lambda}$ and integrate it by parts over C. By L-periodicity, one obtains

$$-\int_C \left[\frac{\nabla\phi_{B,\lambda}A\nabla\phi_{B,\lambda}}{\phi^2_{B,\lambda}}\right] - (\lambda^2 eAe + \mu_0)|C| - B\int_C \nu(x)dx = k_\lambda(B)|C|,$$

and, since $\phi_{B,\lambda}$ is not constant (because ν is not constant) and the matrix A is elliptic, one gets that $k_\lambda(B) < -(\lambda^2 eAe + \mu_0) = k_\lambda(0)$ for all $B > 0$. Hence, since $k'_\lambda(0) = 0$ and $k_\lambda(B)$ is concave in B, one concludes that $B \mapsto k_\lambda(B)$ is decreasing in $B > 0$. Finally, it follows that $c^*(B)$ is increasing in $B > 0$. □

The biological interpretation of this proposition is that increasing the amplitude of the favorableness of the environment increases the invasion's speed.

Remark 4.9 *If one only assumes that* $\max \nu > 0$*, then the function* f *satisfies the hypothesis for conservation for* $B > 0$ *large enough. Furthermore, under the other assumptions of Proposition 4.8, the same arguments as above imply that the function* $B \mapsto c^*(B)$ *is an increasing function of* B *(for* B *large, as soon as the hypothesis for conservation is satisfied).*

In the next proposition, one assumes that f satisfies the hypothesis for conservation. As one has said above, it follows from Theorem 2.8 of [8] that it is true for all $B > 0$ if $\int_C \mu(x) \geq 0$ and $\int_C \nu(x) \geq 0$ with $\nu \not\equiv 0$; if one only has $\max \nu > 0$, it is true if B is large enough.

Proposition 4.10 *Assume that* $\max \nu > 0$ *and that the function* f *satisfies the hypothesis for conservation with* $f_u(x, 0) = \mu(x) + B\nu(x)$*. Then*

$$c^*(B) \leq 2\sqrt{eAe \max(\mu + B\nu)}.$$

PROOF. Let us first observe that the definition of λ_1 in (1.9) and that the hypothesis for conservation ($\lambda_1 < 0$) imply that $\max(\mu + B\nu) > 0$.

Next, using the characterization of c^* given by (4.69), let us integrate by parts over C the equation $-L_{c,B,\lambda}\psi = \mu$. Using the L-periodicity of ψ, one obtains the following inequalities :

$$\mu_{c,B}(\lambda) \geq -\lambda^2 eAe + c\lambda - \max(\mu + B\nu).$$

Therefore, if $c \geq 2\sqrt{eAe \max(\mu + B\nu)}$, there exists $\lambda_0 > 0$ such that $\mu_{c,B}(\lambda_0) \geq 0$. On the other hand, $\mu_{c,B}(0) = \lambda_1 < 0$ from the hypothesis for conservation. By continuity, it follows that there exists a solution $\lambda > 0$ of $\mu_{c,B}(\lambda) = 0$ as soon as $c \geq 2\sqrt{eAe \max(\mu + B\nu)}$. Thus, one finally has

$$c^*(B) \leq 2\sqrt{eAe \max(\mu + B\nu)}. \tag{4.73}$$

That completes the proof of Proposition 4.10. □

Remark 4.11 *If the diffusion matrix field* A *is not assumed to be uniform in the space variables anymore (but still satisfies (1.6)), and if* $\max \nu > 0$*, then the hypothesis for conservation is satisfied for* B *large enough and the same arguments as above imply that*

$$\limsup_{B \to +\infty} \frac{c^*(B)}{\sqrt{B}} \leq 2\sqrt{\max(eAe)}\sqrt{\max \nu}.$$

Proposition 4.12 *Assume now that* $f_u(x,0) = \mu(x) + B\nu(x)$*, where* $\displaystyle\int_C \mu \geq 0$*,* $\displaystyle\int_C \nu \geq 0$ *and* $\max \nu > 0$*. Then*

$$2\sqrt{\frac{eAe}{|C|}\int_C \left(\frac{\mu}{B} + \nu\right) dx} \leq \frac{c^*(B)}{\sqrt{B}} \leq 2\sqrt{eAe \max\left(\frac{\mu}{B} + \nu\right)} \tag{4.74}$$

and

$$\frac{1}{2}\sqrt{eAe \max \nu} \leq \liminf_{B\to+\infty} \frac{c^*(B)}{\sqrt{B}} \leq \liminf_{B\to+\infty} \frac{c^*(B)}{\sqrt{B}} \leq 2\sqrt{eAe \max \nu}. \tag{4.75}$$

PROOF. As already underlined, the assumptions of Proposition 4.12 guarantee that the hypothesis for conservation is satisfied for all $B > 0$.

We will now use the characterization of c^* given by (4.70). Let $\phi_{B,\lambda}$ be defined by (4.71). Dividing (4.71) by $\lambda\phi_{B,\lambda}|C|$ and integrating by parts leads to

$$\lambda eAe + \frac{\displaystyle\int_C (\mu + B\nu)}{\lambda|C|} \leq -\frac{k_\lambda(B)}{\lambda}. \tag{4.76}$$

One deduces from (4.76) and (4.70) that

$$2\sqrt{\frac{eAe}{|C|}\int_C (\mu + B\nu)} \leq c^*(B), \tag{4.77}$$

and the result (4.74) follows from (4.73) and (4.77).

The proof of the lower bound in (4.75) is divided in two steps. Let

$$0 \leq \gamma := \liminf_{B\to+\infty} \frac{c^*(B)}{\sqrt{B}} \leq 2\sqrt{eAe \max \nu}$$

and $(B_n)_{n\in\mathbb{N}} \to +\infty$ such that $c^*(B_n)/\sqrt{B_n} \to \gamma$ as $n \to +\infty$. First, from (4.70), there exits a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^*_+$ such that $\dfrac{-k_{\lambda_n}(B)}{\lambda_n\sqrt{B_n}} \to \gamma$ as $n \to +\infty$. Moreover, from (4.73), one knows that

$$\frac{-k_{\lambda_n}(B)}{\lambda_n\sqrt{B_n}} \leq 2\sqrt{eAe \max \nu} + \varepsilon_n, \tag{4.78}$$

where $\varepsilon_n \to 0$ as $n \to +\infty$. Using the equation (4.76), one obtains with (4.78)

$$\lambda_n eAe + \frac{\displaystyle\int_C (\mu + B\nu)}{|C|\lambda_n} \leq \frac{-k_{\lambda_n}(B_n)}{\lambda_n} \leq 2\sqrt{B_n eAe \max \nu} + \varepsilon_n\sqrt{B_n}.$$

Assuming that $\displaystyle\int_C \mu \geq 0$ and $\displaystyle\int_C \nu \geq 0$, one deduces that

$$\lambda_n \leq 2\sqrt{\frac{B_n}{eAe}\max \nu} + \varepsilon_n\sqrt{B_n}. \tag{4.79}$$

Next, consider the eigenvalue problem

$$\begin{cases} -\nabla\cdot(A\nabla\psi_{B,\lambda}) - 2\lambda Ae\cdot\nabla\psi_{B,\lambda} \\ \qquad -\lambda^2 eAe\psi_{B,\lambda} - (\mu + B\nu)\psi_{B,\lambda} = \widetilde{k}_\lambda(B)\psi_{B,\lambda}, \\ \psi_{B,\lambda} > 0 \text{ on } C,\ \psi_{B,\lambda} = 0 \text{ on } \partial C,\ \|\psi_{B,\lambda}\|_\infty = 1, \end{cases}$$

and let us prove that $\widetilde{k}_\lambda(B) > k_\lambda(B)$ (for all $\lambda > 0$ and $B > 0$). Assume that, on the contrary, one has $\widetilde{k}_\lambda(B) \leq k_\lambda(B)$; then the function $\psi_{B,\lambda}$ satisfies

$$\begin{aligned} -\nabla\cdot(A\nabla\psi_{B,\lambda}) - 2\lambda Ae\cdot\nabla\psi_{B,\lambda} - \lambda^2 eAe\psi_{B,\lambda}\ & -(\mu + B\nu)\psi_{B,\lambda} - k_\lambda(B)\psi_{B,\lambda} \\ & = (\widetilde{k}_\lambda(B) - k_\lambda(B))\psi_{B,\lambda} \leq 0. \end{aligned} \tag{4.80}$$

Since the function $\phi_{B,\lambda}$ defined by (4.71) is positive in $\overline{C}$, one can assume that $\kappa\psi_{B,\lambda} < \phi_{B,\lambda}$ in $\overline{C}$ for all $\kappa > 0$ small enough. Now, set

$$\kappa^* = \sup\left\{\kappa > 0,\ \kappa\psi_{B,\lambda} < \phi_{B,\lambda} \text{ in } \overline{C}\right\} > 0.$$

Then, by continuity, $\kappa^*\psi_{B,\lambda} \leq \phi_{B,\lambda}$ in $\overline{C}$ and there exists x_1 in $\overline{C}$ such that $\kappa^*\psi_{B,\lambda}(x_1) = \phi_{B,\lambda}(x_1)$. But, since $\phi_{B,\lambda} > 0$ in $\overline{C}$ and $\psi_{B,\lambda} = 0$ on ∂C, it follows that $x_1 \in C$. Therefore, using (4.80), it follows from the strong elliptic maximum principle that $\kappa^*\psi_{B,\lambda} \equiv \phi_{B,\lambda}$ in $\overline{C}$, which is impossible from the boundary conditions on ∂C. Finally, one concludes that $\widetilde{k}_\lambda(B) > k_\lambda(B)$.

Let us now define $\Psi_{B,\lambda}(x) = e^{\lambda x\cdot e}\psi_{B,\lambda}(x)$. From (4.80), the function $\Psi_{B,\lambda}$ satisfies the eigenvalue problem

$$\begin{cases} -\nabla\cdot(A\nabla\Psi_{B,\lambda}) - (\mu + B\nu)\Psi_{B,\lambda} = \widetilde{k}_\lambda(B)\Psi_{B,\lambda}, \\ \Psi_{B,\lambda} > 0 \text{ on } C,\ \Psi_{B,\lambda} = 0 \text{ on } \partial C, \end{cases}$$

and it follows that

$$\widetilde{k}_\lambda(B) = \min_{\psi\in H_0^1(C),\ \psi\not\equiv 0} \frac{\displaystyle\int_C \nabla\psi\cdot(A\nabla\psi) - (\mu(x) + B\nu(x))\psi^2}{\displaystyle\int_C \psi^2}.$$

Let $\varepsilon > 0$ be arbitrarily chosen. Then, there exists ψ_ε in $H_0^1(C)$, such that $\|\psi_\varepsilon\|_\infty = 1$, $\psi_\varepsilon \geq 0$ and, for all $x \in C$,

$$\psi_\varepsilon(x) > 0 \Rightarrow (\max\nu\ - \nu(x) < \varepsilon).$$

One then easily obtains (see the proof of Proposition 5.2 in [8])

$$\frac{\displaystyle -\int_C \nabla\psi_\varepsilon\cdot(A(x)\nabla\psi_\varepsilon) + \int_C \mu(x)\psi_\varepsilon^2}{\displaystyle\int_C \psi_\varepsilon^2} + B(\max\nu - \varepsilon) \leq -\widetilde{k}_\lambda(B) \tag{4.81}$$

for all $\lambda > 0$.

Hence, using (4.79) and (4.81), and since $\widetilde{k}_{\lambda_n}(B_n) > k_{\lambda_n}(B_n)$ one has

$$\liminf_{n\to+\infty} \frac{-k_{\lambda_n}(B_n)}{\lambda_n\sqrt{B_n}} \geq \frac{1}{2}\sqrt{eAe}\left(\sqrt{\max\nu} - \frac{\varepsilon}{\sqrt{\max\nu}}\right).$$

since $-k_{\lambda_n}(B) \geq 0$. Since $\epsilon > 0$ was arbitrary, one concludes that

$$\gamma = \liminf_{B\to+\infty} \frac{c^*(B)}{\sqrt{B}} \geq \frac{1}{2}\sqrt{eAe\max\nu}.$$

The formula (4.75) follows. $\square$

PROOF of Corollary 1.4. In the special case where $f_u(x,0) = \mu(x)$ ($\nu = 0$ and, say, $B = 1$) with

$$\int_C \mu(x)dx \geq \mu_0|C| > 0,$$

it follows from the lower bound in (4.74) that $c^*[\mu] \geq c^*[\mu_0] = 2\sqrt{eAe\ \mu_0}$. $\square$

In other words, an heterogeneous medium increases the biological invasion's speed, in comparison with a constant medium, when $\displaystyle\int_C f_u(x,0)dx > 0$.

Coming back to the case where $f_u(x,0) = \mu(x) + B\nu(x)$, it follows from the Proposition 4.12 that, even if μ and ν have zero average, it suffices for ν to be positive somewhere for the speed $c^*(B)$ to increase like to the square root of the amplitude of the effective birth rate.

Proposition 4.13 *Assume that $\mu \equiv 0$, $f_u(x,0) = B\nu(x)$ with $\displaystyle\int_C \nu \geq 0$ and $\max\nu > 0$. Then, one has*

$$\lim_{B\to 0^+} \frac{c^*(B)}{\sqrt{B}} = 2\sqrt{\frac{eAe}{|C|}\int_C \nu(x)dx}.$$

PROOF. First, it follows from (4.74) that

$$2\sqrt{\frac{eAe}{|C|}\int_C \nu(x)dx} \leq \frac{c^*(B)}{\sqrt{B}} \tag{4.82}$$

for all $B > 0$.

In order to establish the opposite inequality at the limit $B \to 0^+$, one shall consider two cases :

Case 1 : $\displaystyle\int_C \nu > 0$. Let $\phi_{B,\lambda}$ be defined by (4.71) and call $\phi_B = \phi_{B,\lambda_B}$ with

$$\lambda_B = \sqrt{\frac{B}{eAe|C|}\int_C \nu(x)dx}.$$

Multiply (4.71) by ϕ_B and integrate it by parts over C. Dividing by $\displaystyle\int_C \phi_B^2$, one obtains

$$k_{\lambda_B}(B) = \frac{\displaystyle\int_C \nabla\phi_B A \nabla\phi_B}{\displaystyle\int_C \phi_B^2} - \lambda_B^2 eAe - B\frac{\displaystyle\int_C \nu\phi_B^2}{\displaystyle\int_C \phi_B^2},$$

and

$$\frac{-k_{\lambda_B}(B)}{\lambda_B} \leq \lambda_B eAe + \frac{B}{\lambda_B}\frac{\displaystyle\int_C \nu\phi_B^2}{\displaystyle\int_C \phi_B^2}.$$

Moreover, observing that $\lambda_B \to 0$ as $B \to +\infty$ and arguing as in the proof of Lemma 3.1, one knows that ϕ_B converges in $C^{2,\beta}(\mathbb{R}^N)$ (for all $0 \leq \beta < 1$) to $\phi_0 \equiv 1$ as $B \to 0$. Therefore, one can write

$$\frac{-k_{\lambda_B}(B)}{\lambda_B} \leq \lambda_B eAe + \frac{B}{\lambda_B|C|}\int_C \nu \ + \frac{B}{\lambda_B}\varepsilon_B, \tag{4.83}$$

where $\varepsilon_B \to 0$ as $B \to 0$. Replacing λ_B by its value in (4.83), one obtains

$$\frac{-k_{\lambda_B}(B)}{\lambda_B} \leq 2\sqrt{eAe\frac{B}{|C|}\int \nu} + \sqrt{\frac{BeAe|C|}{\displaystyle\int_C \nu}}\ \varepsilon_B.$$

¿From the characterization (4.70) of $c^*(B)$, one then obtains

$$\frac{c^*(B)}{\sqrt{B}} \leq 2\sqrt{\frac{eAe}{|C|}\int_C \nu} + \widetilde{\varepsilon}_B,$$

where $\widetilde{\varepsilon}_B \to 0$ as $B \to 0$. Using (4.82), one concludes that

$$\lim_{B\to 0}\frac{c^*(B)}{\sqrt{B}} = 2\sqrt{\frac{eAe}{|C|}\int_C \nu}.$$

Case 2 : $\displaystyle\int_C \nu = 0$. Choose now $\lambda_B = \sqrt{\delta B}$ for arbitrary $\delta > 0$. The above arguments imply that

$$\limsup_{B\to 0^+}\frac{c^*(B)}{\sqrt{B}} \leq \sqrt{\delta}\ eAe$$

and the conclusion follows. □

Finally, Theorem 1.3 follows from the last four propositions.

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